

Planar Traveling Waves For Nonlocal Dispersion Equation With Monostable Nonlinearity

Rui Huang^a, Ming Mei^{b,c} and Yong Wang^d

^a*School of Mathematical Sciences, South China Normal University
Guangzhou, Guangdong, 510631, China
huang@scnu.edu.cn*

and

^b*Department of Mathematics, Champlain College Saint-Lambert
Quebec, J4P 3P2, Canada*

^c*Department of Mathematics and Statistics, McGill University
Montreal, Quebec, H3A 2K6, Canada
ming.mei@mcgill.ca*

and

^d*Institute of Applied Mathematics, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing, 100190, China
yongwang@amss.ac.cn*

Abstract

In this paper, we study a class of nonlocal dispersion equation with monostable nonlinearity in n -dimensional space

$$\begin{cases} u_t - J * u + u + d(u(t, x)) = \int_{\mathbb{R}^n} f_\beta(y) b(u(t - \tau, x - y)) dy, \\ u(s, x) = u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n, \end{cases}$$

where the nonlinear functions $d(u)$ and $b(u)$ possess the monostable characters like Fisher-KPP type, $f_\beta(x)$ is the heat kernel, and the kernel $J(x)$ satisfies $\hat{J}(\xi) = 1 - \mathcal{K}|\xi|^\alpha + o(|\xi|^\alpha)$ for $0 < \alpha \leq 2$. After establishing the existence for both the planar traveling waves $\phi(x \cdot \mathbf{e} + ct)$ for $c \geq c_*$ (c_* is the critical wave speed) and the solution $u(t, x)$ for the Cauchy problem, as well as the comparison principles, we prove that, all noncritical planar wavefronts $\phi(x \cdot \mathbf{e} + ct)$ are globally stable with the exponential convergence rate $t^{-n/\alpha} e^{-\mu_\tau}$ for $\mu_\tau > 0$, and the critical wavefronts $\phi(x \cdot \mathbf{e} + c_* t)$ are globally stable in the algebraic form $t^{-n/\alpha}$. The adopted approach is Fourier transform and the weighted energy method with a suitably selected weight function. These rates are optimal and the stability results significantly develop the existing studies for nonlocal dispersion equations.

Keywords: Nonlocal dispersion equations, traveling waves, global stability, the Fisher-KPP equation, time-delays, weighted energy, Fourier transform.

AMS: 35K57, 34K20, 92D25

Contents

1	Introduction	2
2	Traveling Waves and Their Stabilities	7
3	Linearized Nonlocal dispersion Equations	10
4	Global Existence and Comparison principle	16
5	Global Stability of Planar Traveling Waves	19
6	Applications and Concluding Remark	26
6.1	Nicholson's blowflies equation with nonlocal dispersion	26
6.2	Fisher-KPP equation with nonlocal dispersion	27
6.3	Concluding Remark	28

1 Introduction

For the gradient flow to an order parameter describing the state of a solid material, for example, a perfect crystal with two different orientations, it is usually described by a convolution model of phase transition in the form [2, 4, 8, 23, 24]

$$u_t = J * u - u + F(u), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad (1.1)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $J(x)$ is a non-negative and radial kernel with unit integral, and

$$(J * u)(t, x) = \int_{\mathbb{R}^n} J(x - y)u(t, y)dy. \quad (1.2)$$

As showed in [4, 11], when the kernel $J(x)$ has a second momentum, for example, J is compact-supported or Gaussian-like kernel $J \sim e^{-x^2}$, its Fourier transform looks like

$$\hat{J}(\xi) = 1 - \mathcal{K}|\xi|^2 + o(|\xi|^2), \quad \mathcal{K} > 0,$$

then the effect of the nonlocal dispersion $J * u - u$ is almost the same to the linear diffusion $\mathcal{K}\Delta u$:

$$J * u - u \approx \mathcal{K}\Delta u,$$

which informs us to expect that the behaviors of the solutions to the nonlocal dispersion equation and the linear diffusion equation are almost identical [4, 8, 23, 24]

$$u_t = J * u - u \Leftrightarrow u_t = \mathcal{K}\Delta u.$$

Notice that, comparing with the heat equations, the solutions for the nonlocal dispersion equations usually loss the spatial regularity, but have much better regularity in time, see Remark 4.2 below for details.

In general, $J(x)$ may not have a second momentum, let us say,

$$\hat{J}(\xi) = 1 - \mathcal{K}|\xi|^\alpha + o(|\xi|^\alpha) \quad \text{as } \xi \rightarrow 0 \quad \text{for } \alpha \in (0, 2).$$

One example is the Cauchy law by taking $J(x) = \frac{1}{1+|x|^2}$ which implies its Fourier transform mentioned above with $\alpha = 1$. In this case, the behavior of the solutions to the nonlocal dispersion equation is almost identical to the fractional diffusion equation [4, 8, 23, 24]

$$u_t = J * u - u \Leftrightarrow u_t = \mathcal{K} \Delta^{\alpha/2} u.$$

Equation (1.1) represents also the dynamical population model of single species in ecology [13], where $u(t, x)$ is the density of population at location x and time t , and $J(x - y)$ is thought of as the probability distribution of jumping from location y to location x , and $J * u = \int_{\mathbb{R}^n} J(x - y)u(t, y)dy$ is the rate at which individuals are arriving to position x from all other places, while $-u(x, t) = -\int_{\mathbb{R}^n} J(x - y)u(t, x)dy$ stands the rate at which they are leaving the location x to travel to all other places. In this case, under the consideration of the effects from birth rate and death rate, the equation (1.1) is usually written as follows

$$u_t = J * u - u + b(u(t - \tau, x)) - d(u(t, x)), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad (1.3)$$

where $b(u(t - \tau, x))$ is the birth rate function, $d(u(t, x))$ is the death rate function, and $\tau > 0$ is the mature age of the single species, which is usually called the *time-delay*. Furthermore, if we consider the distribution of all matured population, the effect of birth rate is then involved in whole space \mathbb{R}^n [20, 39, 47], and the equation is expressed as

$$\frac{\partial u}{\partial t} - J * u + u + d(u(t, x)) = \int_{\mathbb{R}^n} f_\beta(y) b(u(t - \tau, x - y)) dy, \quad (1.4)$$

where $f_\beta(y)$, with $\beta > 0$, is the heat kernel in the form of

$$f_\beta(y) = \frac{1}{(4\pi\beta)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4\beta}} \quad \text{with} \quad \int_{\mathbb{R}^n} f_\beta(y) dy = 1. \quad (1.5)$$

Notice that, by using the property of heat kernel

$$\lim_{\beta \rightarrow 0^+} \int_{\mathbb{R}^n} f_\beta(y) b(u(t - \tau, x - y)) dy = b(u(t - \tau, x)),$$

we then derive the equation (1.3) as a limit of the equation (1.4) by taking $\beta \rightarrow 0^+$, and further derive the regular nonlocal dispersion equation (1.1) from the equation (1.3) by taking the time-delay $\tau = 0$ and $F(u) = b(u) - d(u)$. In particular, if we set $d(u) = u^2$ and $b(u) = u$, then, from (1.1) we get the classical Fisher-KPP equation with nonlocal dispersion

$$u_t = J * u - u + u(1 - u). \quad (1.6)$$

So, the equations (1.1) and (1.3) and (1.6) all are the special cases of the equation (1.4).

In this paper, we will concentrate ourselves to the Cauchy problem for the more generalized equation (1.4) with non-locality of birth rate

$$\begin{cases} \frac{\partial u}{\partial t} - J * u + u + d(u(t, x)) = \int_{\mathbb{R}^n} f_\beta(y) b(u(t - \tau, x - y)) dy, \\ u(s, x) = u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n. \end{cases} \quad (1.7)$$

When $\tau = 0$ (no time-delay), then the above equation is reduced to

$$\begin{cases} \frac{\partial u}{\partial t} - J * u + u + d(u) = \int_{\mathbb{R}^n} f_\beta(y) b(u(t, x - y)) dy, \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \quad (1.8)$$

We will also discuss how the time-delay τ effects the property of the solutions.

For the equation (1.1) in 1D case, when $F(u)$ is bistable, namely, two constant equilibria u_- and u_+ both are the stable nodes (the typical example is the Huxley equation with $F(u) = u(u - a)(1 - u)$ for $0 < a < 1$), Bates *et al* [2] and Chen [6] proved that the traveling waves are globally stable as $t \rightarrow +\infty$. In this paper, we consider another important type of equations with monostable nonlinearity. The typical example in this case is Fisher-KPP equation with $F(u) = u(1 - u)$. Hence, throughout this paper, we assume that the death rate $d(u)$ and birth rate $b(u)$ capture the following characters of monostable nonlinearity:

(H₁) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(u), b(u) \in C^2[0, u_+]$;

(H₂) $b'(0) > d'(0) \geq 0$ and $0 \leq b'(u_+) < d'(u_+)$;

(H₃) For $0 \leq u \leq u_+$, $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$, $b''(u) \leq 0$.

These characters are summarized from the classical Fisher-KPP equation, see also the monostable reaction-diffusion equations in ecology, for example, the Nicholson's blowflies equation [37, 38, 39, 47] with

$$d(u) = \delta u \quad \text{and} \quad b(u) = pue^{-au}, \quad p > 0, \delta > 0, a > 0$$

and $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{p}{\delta} > 0$ under the consideration of $1 < \frac{p}{\delta} \leq e$; and the age-structured population model [19, 20, 39, 42, 44] with

$$d(u) = \delta u^2 \quad \text{and} \quad b(u) = pe^{-\gamma\tau}u, \quad \delta > 0, p > 0, \gamma > 0,$$

and $u_- = 0$ and $u_+ = \frac{p}{\delta} e^{-\gamma\tau}$.

Clearly, under the hypothesis (H₁)-(H₃), both $u_- = 0$ and $u_+ > 0$ are constant equilibria of the equation (1.7), and $u_- = 0$ is unstable and u_+ is stable for the spatially homogeneous equation associated with (1.7), this is why we call the equation (1.7), including (1.1) and (1.3) and (1.8), as monostable.

On the other hand, we also assume the kernel $J(x)$ satisfying:

(J₁) $J(x) = \prod_{i=1}^n J_i(x_i)$, where $J_i(x_i)$ is smooth, and $J_i(x_i) = J_i(|x_i|) \geq 0$ and $\int_{\mathbb{R}} J_i(x_i) dx_i = 1$ for $i = 1, 2, \dots, n$, and $\int_{\mathbb{R}} |y_1| J_1(y_1) e^{-\lambda_* y_1} dy_1 < \infty$ for $\lambda_* > 0$ defined in (2.3) and (2.4);

(J₂) Fourier transform of $J(x)$ satisfies $\hat{J}(\xi) = 1 - \mathcal{K}|\xi|^\alpha + o(|\xi|^\alpha)$ as $\xi \rightarrow 0$ with $\alpha \in (0, 2]$ and $\mathcal{K} > 0$.

A *planar traveling wavefront* to the equation (1.7) for $\tau \geq 0$ is a special solution in the form of $u(t, x) = \phi(x \cdot \mathbf{e} + ct)$ with $\phi(\pm\infty) = u_\pm$, where c is the wave speed, \mathbf{e} is a unit vector of the basis of \mathbb{R}^n .

Without loss of generality, we can always assume $\mathbf{e} = \mathbf{e}_1 = (1, 0, \dots, 0)$ by rotating the coordinates. Thus, the planar traveling wavefront $\phi(x \cdot \mathbf{e}_1 + ct) = \phi(x_1 + ct)$ satisfies, for $\tau \geq 0$,

$$\begin{cases} c\phi' - J * \phi + \phi + d(\phi) = \int_{\mathbb{R}^n} f_\beta(y) b(\phi(\xi_1 - y_1 - c\tau)) dy, \\ \phi(\pm\infty) = u_\pm, \end{cases} \quad (1.9)$$

where $' = \frac{d}{d\xi_1}$ and $\xi_1 = x_1 + ct$. Let

$$f_{i\beta}(y_i) := \frac{1}{(4\pi\beta)^{1/2}} e^{-\frac{y_i^2}{4\beta}}. \quad (1.10)$$

Then

$$f_\beta(y) := \prod_{i=1}^n f_{i\beta}(y_i), \quad \text{and} \quad \int_{\mathbb{R}} f_{i\beta}(y_i) dy_i = 1, \quad i = 1, 2, \dots, n, \quad (1.11)$$

and (1.9) is reduced to, for $\tau \geq 0$,

$$\begin{cases} c\phi' - J_1 * \phi + \phi + d(\phi) = \int_{\mathbb{R}} f_{1\beta}(y_1) b(\phi(\xi_1 - y_1 - c\tau)) dy_1, \\ \phi(\pm\infty) = u_\pm. \end{cases} \quad (1.12)$$

The main purpose of this paper is to study the global asymptotic stability of planar traveling wavefronts of the equations (1.7) and (1.8) with or without time-delay, respectively, in particular, in the case of the *critical wave* $\phi(x_1 + c_*t)$. Here the number c_* is called the *critical speed* (or the *minimum speed*) in the sense that a traveling wave $\phi(x_1 + ct)$ exists if $c \geq c_*$, while no traveling wave $\phi(x_1 + ct)$ exists if $c < c_*$.

The nonlocal dispersion equation (1.1) has been extensively studied recently. Chasseigne *et al* [4] and Cortazar *et al* [8] showed that the linear nonlocal dispersion equation (1.1) (with $F = 0$) is almost equivalent to the linear diffusion equation, and the asymptotic behavior of the solutions to the linear equation of nonlocal dispersion is exactly the same to the corresponding linear diffusion equation. Ignat and Rossi [23, 24] further obtained the asymptotic behavior of the solutions to the nonlinear equation (1.1). García-Melián and Quirós [17] investigated the blow up phenomenon of the solution to the equation (1.1) with $F(u) = u^p$, and gave the Fujita critical exponent. Regarding the structure of special solutions to (1.1) like traveling wave solutions, early in 1997 Bates *et al* [2] and Chen [6] established the existence of the traveling waves for (1.1) with bistable nonlinearity, and proved their global stability. For (1.1) with monostable nonlinearity, recently Coville and his collaborators [9, 10, 11, 12] studied the existence and uniqueness (up to a shift) of traveling waves. See also the existence/nonexistence of traveling waves by Yagisita [53] and the existence of almost periodic traveling waves by Chen [5]. However, the stability of traveling waves for the nonlocal equation (1.7) (including (1.1) and (1.3)) with monostable nonlinearity is almost not related, except a special case for the fast waves with large wave speed to the 1D age-structured population model by Pan *et al* [44]. As we know, such a problem is also very significant but challenging, because the equations of Fisher-KPP type possess an unstable node, different from the bistable case, this unstable node usually causes a serious difficulty in the stability proof, particularly, for the critical traveling waves. The main interest in this paper is to investigate the stability of traveling waves to (1.7) with $\tau > 0$ and (1.8) with $\tau = 0$. An easy to follow method will be introduced for the stability proof to the nonlocal dispersion equations.

In this paper, we will first investigate the linearized equation of (1.7), and derive the optimal decay rates of the solution to the linearized equation by means of Fourier transform. This is a crucial step for get the optimal convergence for the nonlocal stability of traveling waves. Then, we will technically establish the global existence and comparison principles of the solution to the n -D nonlinear equation with nonlocal dispersion (1.7). Inspired by [43] for the classical Fisher-KPP equations and the further developments by [39, 40], by ingeniously selecting a weight function which is dependent on the critical wave speed c_* , and using the weighted energy method and the Green function method with the comparison principles together, we will further prove that, all noncritical planar traveling waves $\phi(x \cdot \mathbf{e} + ct)$ are exponentially stable in the form of $t^{-\frac{n}{\alpha}} e^{-\mu_\tau}$ for some constant $\mu_\tau = \mu(\tau)$ such that $0 < \mu_\tau \leq \mu_0$ for $\tau \geq 0$; and all critical planar traveling waves $\phi(x \cdot \mathbf{e} + c_*t)$ are algebraically stable in the form of $t^{-\frac{n}{\alpha}}$. These convergence rates are optimal and the stability results significantly develop the existing studies on the nonlocal dispersion equations. We will also show that the time-delay τ will slow down the convergence of the the solution $u(t, x)$ to the noncritical planar traveling waves $\phi(x \cdot \mathbf{e} + ct)$ with $c > c_*$, and cause the higher requirement for the initial perturbation around the wavefronts.

For the stability of traveling waves to other modeling equations, we refer to the classical and significant contributions in [1, 3, 7, 14, 16, 21, 22, 25, 29, 30, 31, 32, 33, 37, 38, 39, 42, 43, 45, 46, 49, 50, 51, 52] for reaction-diffusion equations and [15, 18, 26, 27, 34, 35, 36, 48, 54] for fluid dynamical systems, and the references therein.

The paper is organized as follows. In section 2, we will state the existence of the traveling waves, and their stability. In section 3, we will give the solution formulas to the linearized dispersion equations of (1.7) and (1.8), and derive the optimal decay rates by Fourier transform with energy method together. In section 4, we will prove the global existence of the solution to (1.7) and establish the comparison principle. In section 5, based on the results obtained in sections 3 and 4, by using the weighted energy method, we will further prove the stability of planar traveling waves including the critical and noncritical waves. Finally, in section 6, we will give some particular applications of our stability theory to the classical Fisher-KPP equation with nonlocal dispersion and the Nicholson's blowflies model, and make a concluding remark to a more general case.

Before ending this section, we make some notations. Throughout this paper, $C > 0$ denotes a generic constant, while $C_i > 0$ and $c_i > 0$ ($i = 0, 1, 2, \dots$) represent specific constants. $j = (j_1, j_2, \dots, j_n)$ denotes a multi-index with non-negative integers $j_i \geq 0$ ($i = 1, \dots, n$), and $|j| = j_1 + j_2 + \dots + j_n$. The derivatives for multi-dimensional function are denoted as

$$\partial_x^j f(x) := \partial_{x_1}^{j_1} \cdots \partial_{x_n}^{j_n} f(x).$$

For a n -D function $f(x)$, its Fourier transform is defined as

$$\mathcal{F}[f](\eta) = \hat{f}(\eta) := \int_{\mathbb{R}^n} e^{-\mathbf{i}x \cdot \eta} f(x) dx, \quad \mathbf{i} := \sqrt{-1},$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}x \cdot \eta} \hat{f}(\eta) d\eta.$$

Let I be an interval, typically $I = \mathbb{R}^n$. $L^p(I)$ ($p \geq 1$) is the Lebesgue space of the integrable functions defined on I , $W^{k,p}(I)$ ($k \geq 0, p \geq 1$) is the Sobolev space of the L^p -functions $f(x)$ defined on the

interval I whose derivatives $\partial_x^j f$ with $|j| = k$ also belong to $L^p(I)$, and in particular, we denote $W^{k,2}(I)$ as $H^k(I)$. Further, $L_w^p(I)$ denotes the weighted L^p -space for a weight function $w(x) > 0$ with the norm defined as

$$\|f\|_{L_w^p} = \left(\int_I w(x) |f(x)|^p dx \right)^{1/p},$$

$W_w^{k,p}(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{W_w^{k,p}} = \left(\sum_{|j|=0}^k \int_I w(x) |\partial_x^j f(x)|^p dx \right)^{1/p},$$

and $H_w^k(I)$ is defined with the norm

$$\|f\|_{H_w^k} = \left(\sum_{|j|=0}^k \int_I w(x) |\partial_x^j f(x)|^2 dx \right)^{1/2}.$$

Let $T > 0$ be a number and \mathcal{B} be a Banach space. We denote by $C^0([0, T], \mathcal{B})$ the space of the \mathcal{B} -valued continuous functions on $[0, T]$, $L^2([0, T], \mathcal{B})$ as the space of the \mathcal{B} -valued L^2 -functions on $[0, T]$. The corresponding spaces of the \mathcal{B} -valued functions on $[0, \infty)$ are defined similarly.

2 Traveling Waves and Their Stabilities

As we mentioned before, the existence and uniqueness (up to a shift) of traveling waves for the equation (1.1) were proved in [9, 10, 11, 12], particular, in a recent work by Yagisita [53] for the existence and nonexistence of traveling waves, when the nonlinearity $F(u)$ is monostable. Without any difficulty, these results can be extended to the nonlocal equation (1.7) with time-delay with the help of comparison principle established in Section 4, when $d(u)$ and $b(u)$ satisfy the monostable features (H_1) -(H_3). We state these results as follows without detailed proof.

Theorem 2.1 *Under the conditions (H_1) -(H_3) and (J_1) -(J_2), for the time-delay $\tau \geq 0$, there exist a minimum wave speed (also called the critical wave speed) $c_* > 0$ such that*

- *when $c \geq c_*$, there exists a monotone traveling wavefront $\phi(x_1 + ct)$ of (1.9) connecting u_{\pm} exists;*
- *when $c < c_*$, no traveling wave $\phi(x_1 + ct)$ exists.*

Here (c_*, λ_*) with $c_* > 0$ and $\lambda_* > 0$ is given by

$$H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*), \quad H'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*), \quad (2.1)$$

where

$$H_c(\lambda) = b'(0)e^{\beta\lambda^2 - \lambda c\tau}, \quad G_c(\lambda) = c\lambda - E_c(\lambda) + d'(0), \quad E_c(\lambda) = \int_{\mathbb{R}} J_1(y_1) e^{-\lambda y_1} dy_1 - 1, \quad (2.2)$$

namely, (c_*, λ_*) is the tangent point of $H_c(\lambda)$ and $G_c(\lambda)$ specified as

$$b'(0)e^{\beta\lambda_*^2 - \lambda_* c_* \tau} = c_* \lambda_* - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda_* y_1} dy_1 + 1 + d'(0), \quad (2.3)$$

$$b'(0)(2\beta\lambda_* - c_* \tau)e^{\beta\lambda_*^2 - \lambda_* c_* \tau} = c_* + \int_{\mathbb{R}} y_1 J_1(y_1) e^{-\lambda_* y_1} dy_1. \quad (2.4)$$

Furthermore, it can be verified:

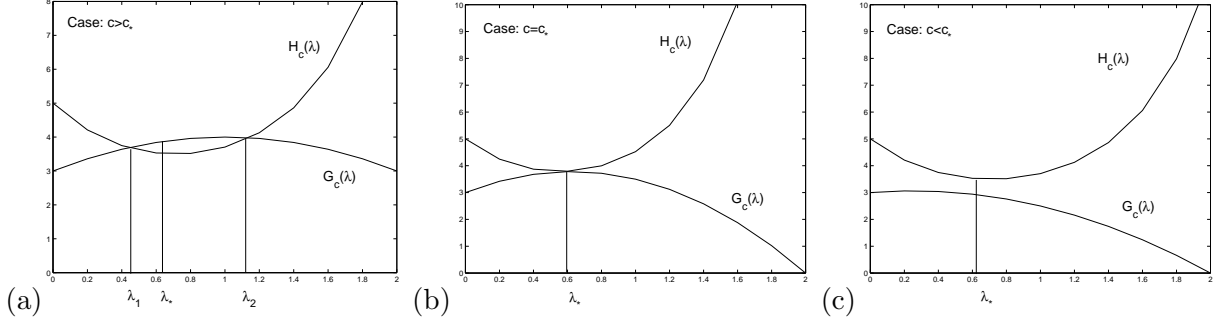


Figure 2.1: (a): the case of $c > c_*$; (b): the case of $c = c_*$; and (c): the case of $c < c_*$.

- In the case of $c > c_*$, there exist two numbers depending on c : $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) > 0$ as the solutions to the equation $H_c(\lambda_i) = G_c(\lambda_i)$, i.e.,

$$b'(0)e^{\beta\lambda_i^2 - \lambda_i c\tau} = c\lambda_i - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda_i y_1} dy_1 + d'(0), \quad i = 1, 2, \quad (2.5)$$

such that

$$H_c(\lambda) < G_c(\lambda) \quad \text{for } \lambda_1 < \lambda < \lambda_2, \quad (2.6)$$

and particularly,

$$H_c(\lambda_*) < G_c(\lambda_*) \quad \text{with } \lambda_1 < \lambda_* < \lambda_2. \quad (2.7)$$

- In the case of $c = c_*$, it holds

$$H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \quad \text{with } \lambda_1 = \lambda_* = \lambda_2. \quad (2.8)$$

- When $\xi_1 = x_1 + ct \rightarrow \pm\infty$, for all $c \geq c_*$, the traveling wavefronts $\phi(x_1 + ct)$ converge to u_{\pm} exponentially as follows

$$|\phi(\xi_1) - u_{\pm}| = O(1)e^{-\lambda^{\pm}|\xi_1|}. \quad (2.9)$$

Here $\lambda^- = \lambda_1(c) > 0$ is given in (2.5), and $\lambda^+ = \lambda^+(c) > 0$ is the unique root determined by the following equation

$$-c\lambda^+ - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda^+ y_1} dy_1 + d'(u_+) = b'(u_+)e^{\beta(\lambda^+)^2 - \lambda^+ c\tau}. \quad (2.10)$$

For easily understanding all cases mentioned in the above, we show them in Figure 2.1.

Before stating our main stability theorems, let us technically choose a weight function:

$$w(x_1) = \begin{cases} e^{-\lambda_*(x_1 - x_*)}, & \text{for } x_1 \leq x_*, \\ 1, & \text{for } x_1 > x_*, \end{cases} \quad (2.11)$$

where $\lambda_* = \lambda_*(c_*) > 0$ is given in (2.3) and (2.4), and $x_* > 0$ is a sufficiently large number such that,

$$0 < d'(\phi(x_*)) - \int_{\mathbb{R}^n} f_{\beta}(y)b'(\phi(x_* - y_1 - c\tau))dy < d'(u_+) - b'(u_+). \quad (2.12)$$

The selection of x_* in (2.12) is valid, because of $d'(u_+) - b'(u_+) > 0$ (see(H₂)). In fact, we have

$$\lim_{\xi_1 \rightarrow \infty} d'(\phi(\xi_1)) = d'(u_+)$$

$$\begin{aligned}
&> b'(u_+) \\
&= \int_{\mathbb{R}^n} f_\beta(y) \left[\lim_{\xi_1 \rightarrow \infty} b'(\phi(\xi_1 - y_1 - c\tau)) \right] dy \\
&= \lim_{\xi_1 \rightarrow \infty} \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy,
\end{aligned}$$

which implies that, by (H_3) , there exists a unique $x_* \gg 1$ such that, for $\xi_1 \in [x_*, \infty)$

$$\begin{aligned}
&d'(u_+) - b'(u_+) \\
&> d'(\phi(\xi_1)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy \\
&\geq d'(\phi(x_*)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(x_* - y_1 - c\tau)) dy \\
&> 0.
\end{aligned} \tag{2.13}$$

Theorem 2.2 (Stability of planar traveling waves with time-delay) *Under assumptions (H_1) - (H_3) and (J_1) - (J_2) , for a given traveling wave $\phi(x_1 + ct)$ of the equation (1.7) with $c \geq c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data $u_0(s, x)$ is bounded in $[u_-, u_+]$ and $u_0 - \phi \in C([- \tau, 0]; H_w^m(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ and $\partial_s(u_0 - \phi) \in L^1([- \tau, 0]; H_w^m(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ with $m > \frac{n}{2}$, then the solution of (1.7) uniquely exists and satisfies:*

- When $c > c_*$, the solution $u(t, x)$ converges to the noncritical planar traveling wave $\phi(x_1 + ct)$ exponentially

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad t > 0, \tag{2.14}$$

where

$$0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}, \tag{2.15}$$

and $\varepsilon_1 = \varepsilon_1(\tau)$ such that $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $\varepsilon_1 = \varepsilon_1(\tau) \rightarrow 0^+$ as $\tau \rightarrow +\infty$;

- When $c = c_*$, the solution $u(t, x)$ converges to the critical planar traveling wave $\phi(x_1 + c_*t)$ algebraically

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_*t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \tag{2.16}$$

However, when the time-delay $\tau = 0$, then we have the following stronger stability for the traveling waves but with a weaker condition on initial perturbation.

Theorem 2.3 (Stability of planar traveling waves without time-delay) *Under assumptions (H_1) - (H_3) and (J_1) - (J_2) , for a given traveling wave $\phi(x_1 + ct)$ of the equation (1.8) with $c \geq c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data $u_0(x)$ is bounded in $[u_-, u_+]$ and $u_0 - \phi \in H_w^m(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n)$ with $m > \frac{n}{2}$, then the solution of (1.8) uniquely exists and satisfies:*

- When $c > c_*$, the solution $u(t, x)$ converges to the noncritical planar traveling wave $\phi(x_1 + ct)$ exponentially

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad t > 0, \tag{2.17}$$

where

$$0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}; \tag{2.18}$$

- When $c = c_*$, the solution $u(t, x)$ converges to the critical planar traveling wave $\phi(x_1 + c_*t)$ algebraically

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_*t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \quad (2.19)$$

Remark 2.4

1. Comparing Theorem 2.2 with time-delay and Theorem 2.3 without time-delay, we realize that, the sufficient condition on the initial perturbation around the wave in the case with time-delay is stronger than the case without time-delay, but the convergence rate to the noncritical waves $\phi(x_1 + ct)$ for $c > c_*$ in the case with time-delay is weaker than the case without time-delay, see (2.15) for $\mu_\tau \leq \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)] < G_c(\lambda_*) - H_c(\lambda_*)$, and (2.18) for $\mu_0 \leq G_c(\lambda_*) - H_c(\lambda_*)$, and $\varepsilon_1 = \varepsilon_1(\tau) \rightarrow 0^+$ as $\tau \rightarrow +\infty$. This means, the time-delay $\tau > 0$ effects the stability of traveling waves a lot, not only the higher requirement for the initial perturbation, but also the slower convergence rate for the solution to the noncritical traveling waves.
2. The convergence rates showed both in Theorem 2.2 and Theorem 2.3 are explicit and optimal, particularly, the algebraic decay rates for the solution converging to the critical waves. Actually, all of them are derived from the linearized equations.

3 Linearized Nonlocal dispersion Equations

In this section, we will derive the solution formulas for the linearized nonlocal dispersion equations with or without time-delay, as well as their optimal decay rates, which will play a key role in the stability proof in section 5.

Now let us introduce the solution formula for linear delayed ODEs [28] and the asymptotic behaviors of the solutions [41].

Lemma 3.1 ([28]) *Let $z(t)$ be the solution to the following linear time-delayed ODE with time-delay $\tau > 0$*

$$\begin{cases} \frac{d}{dt}z(t) + k_1z(t) = k_2z(t - \tau) \\ z(s) = z_0(s), \quad s \in [-\tau, 0]. \end{cases} \quad (3.1)$$

Then

$$z(t) = e^{-k_1(t+\tau)} e^{\bar{k}_2 t} z_0(-\tau) + \int_{-\tau}^0 e^{-k_1(t-s)} e^{\bar{k}_2(t-\tau-s)} [z'_0(s) + k_1 z_0(s)] ds, \quad (3.2)$$

where

$$\bar{k}_2 := k_2 e^{k_1 \tau}, \quad (3.3)$$

and $e_{\tau}^{\bar{k}_2 t}$ is the so-called delayed exponential function in the form

$$e_{\tau}^{\bar{k}_2 t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 + \frac{\bar{k}_2 t}{1!}, & 0 \leq t < \tau, \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots & \vdots \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-\tau)^2}{2!} + \cdots + \frac{\bar{k}_2^m [t-(m-1)\tau]^m}{m!}, & (m-1)\tau \leq t < m\tau, \\ \vdots & \vdots \end{cases} \quad (3.4)$$

and $e_{\tau}^{\bar{k}_2 t}$ is the fundamental solution to

$$\begin{cases} \frac{d}{dt} z(t) = \bar{k}_2 z(t - \tau) \\ z(s) \equiv 1, \quad s \in [-\tau, 0]. \end{cases} \quad (3.5)$$

Lemma 3.2 ([41]) *Let $k_1 \geq 0$ and $k_2 \geq 0$. Then the solution $z(t)$ to (3.1) (or equivalently (3.2)) satisfies*

$$|z(t)| \leq C_0 e^{-k_1 t} e_{\tau}^{\bar{k}_2 t}, \quad (3.6)$$

where

$$C_0 := e^{-k_1 \tau} |z_0(-\tau)| + \int_{-\tau}^0 e^{k_1 s} |z_0'(s) + k_1 z_0(s)| ds, \quad (3.7)$$

and the fundamental solution $e_{\tau}^{\bar{k}_2 t}$ with $\bar{k}_2 > 0$ to (3.5) satisfies

$$e_{\tau}^{\bar{k}_2 t} \leq C(1+t)^{-\gamma} e^{\bar{k}_2 t}, \quad t > 0, \quad (3.8)$$

for arbitrary number $\gamma > 0$.

Furthermore, when $k_1 \geq k_2 \geq 0$, there exists a constant $\varepsilon_1 = \varepsilon_1(\tau)$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $\varepsilon_1 = 1$ for $\tau = 0$, and $\varepsilon_1 = \varepsilon_1(\tau) \rightarrow 0^+$ as $\tau \rightarrow +\infty$, such that

$$e^{-k_1 t} e_{\tau}^{\bar{k}_2 t} \leq C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0, \quad (3.9)$$

and the solution $z(t)$ to (3.1) satisfies

$$|z(t)| \leq C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0. \quad (3.10)$$

Now, we consider the following linearized nonlocal time-delayed dispersion equation (which will be derived in section 5 for the proof of stability of traveling wavefronts)

$$\begin{cases} \frac{\partial v}{\partial t} - \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} v(t, x - y) dy + c_1 v \\ \quad = c_2 \int_{\mathbb{R}^n} f_{\beta}(y) e^{-\lambda_* (y_1 + c\tau)} v(t - \tau, x - y) dy, \\ v(s, x) = v_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n \end{cases} \quad (3.11)$$

for some given constant coefficients c , c_1 and c_2 , where $c \geq c_*$ is the wave speed.

We are going to derive its solution formula as well as the asymptotic behavior of the solution. By taking Fourier transform to (3.11), and noting that,

$$\begin{aligned}
& \mathcal{F} \left[\int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} v(t, x - y) dy \right] (t, \eta) \\
&= \int_{\mathbb{R}^n} e^{-i x \cdot \eta} \left(\int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} v(t, x - y) dy \right) dx \\
&= \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} \left(\int_{\mathbb{R}^n} e^{-i x \cdot \eta} v(t, x - y) dx \right) dy \\
&= \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} \left(\int_{\mathbb{R}^n} e^{-i(x+y) \cdot \eta} v(t, x) dx \right) dy \\
&= \left(\int_{\mathbb{R}^n} e^{-i y \cdot \eta} J(y) e^{-\lambda_* y_1} dy \right) \hat{v}(t, \eta),
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
& \mathcal{F} \left[c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} v(t - \tau, x - y) dy \right] (t - \tau, \eta) \\
&= c_2 \int_{\mathbb{R}^n} e^{-i x \cdot \eta} \left(\int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} v(t - \tau, x - y) dy \right) dx \\
&= c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} \left(\int_{\mathbb{R}^n} e^{-i x \cdot \eta} v(t - \tau, x - y) dx \right) dy \\
&= c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} \left(\int_{\mathbb{R}^n} e^{-i(x+y) \cdot \eta} v(t - \tau, x) dx \right) dy \\
&= c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} e^{-i y \cdot \eta} \left(\int_{\mathbb{R}^n} e^{-i x \cdot \eta} v(t - \tau, x) dx \right) dy \\
&= \left(c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} e^{-i y \cdot \eta} dy \right) \hat{v}(t - \tau, \eta),
\end{aligned} \tag{3.13}$$

we have

$$\frac{d\hat{v}}{dt} + A(\eta)\hat{v} = B(\eta)\hat{v}(t - \tau, \eta), \quad \text{with } \hat{v}(s, \eta) = \hat{v}_0(s, \eta), \quad s \in [-\tau, 0], \tag{3.14}$$

where

$$A(\eta) := c_1 - \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} e^{-i y \cdot \eta} dy \tag{3.15}$$

and

$$B(\eta) := c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} e^{-i y \cdot \eta} dy. \tag{3.16}$$

By using the formula of the delayed ODE (3.2) in Lemma 3.1, we then solve (3.14) as follows

$$\begin{aligned}
\hat{v}(t, \eta) &= e^{-A(\eta)(t+\tau)} e_{\tau}^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta) \\
&\quad + \int_{-\tau}^0 e^{-A(\eta)(t-s)} e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)} \left[\partial_s \hat{v}_0(s, \eta) + A(\eta) \hat{v}_0(s, \eta) \right] ds,
\end{aligned} \tag{3.17}$$

where

$$\mathcal{B}(\eta) := B(\eta) e^{A(\eta)\tau}. \tag{3.18}$$

Then, by taking the inverse Fourier transform to (3.17), we get

$$v(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i x \cdot \eta} e^{-A(\eta)(t+\tau)} e_{\tau}^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta) d\eta$$

$$\begin{aligned}
& + \int_{-\tau}^0 \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} e^{-A(\eta)(t-s)} e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)} \\
& \times \left[\partial_s \hat{v}_0(s, \eta) + A(\eta) \hat{v}_0(s, \eta) \right] d\eta ds,
\end{aligned} \tag{3.19}$$

and its derivatives

$$\begin{aligned}
\partial_{x_j}^k v(t, x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} (i\eta_j)^k e^{-A(\eta)(t+\tau)} e_{\tau}^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta) d\eta \\
& + \int_{-\tau}^0 \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} (i\eta_j)^k e^{-A(\eta)(t-s)} e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)} \\
& \times \left[\partial_s \hat{v}_0(s, \eta) + A(\eta) \hat{v}_0(s, \eta) \right] d\eta ds
\end{aligned} \tag{3.20}$$

for $k = 0, 1, \dots$ and $j = 1, \dots, n$.

Now we are going to derive the asymptotic behavior of $v(t, x)$.

Proposition 3.3 (Optimal decay rates for $\tau > 0$) *Suppose that $v_0 \in C([- \tau, 0]; H^{m+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$ and $\partial_s v_0 \in L^1([- \tau, 0]; H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$ for $m \geq 0$, and let*

$$\begin{cases} \tilde{c}_1 := c_1 - \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} dy, \\ c_3 := c_2 \int_{\mathbb{R}^n} f_{\beta}(y) e^{-\lambda_*(y_1 + c\tau)} dy > 0. \end{cases} \tag{3.21}$$

If $\tilde{c}_1 \geq c_3$, then there exists a constant $\varepsilon_1 = \varepsilon_1(\tau)$ as showed in (3.9) satisfying $0 < \varepsilon_1 < 1$ for $\tau > 0$, such that the solution of the linearized equation (3.11) satisfies

$$\|\partial_{x_j}^k v(t)\|_{L^2(\mathbb{R}^n)} \leq C \mathcal{E}_{v_0}^k t^{-\frac{n+2k}{2\alpha}} e^{-\varepsilon_1(\tilde{c}_1 - c_3)t}, \quad t > 0, \tag{3.22}$$

for $k = 0, 1, \dots, [m]$ and $j = 1, \dots, n$, where

$$\begin{aligned}
\mathcal{E}_{v_0}^k &:= \|v_0(-\tau)\|_{L^1(\mathbb{R}^n)} + \|v_0(-\tau)\|_{H^k(\mathbb{R}^n)} \\
& + \int_{-\tau}^0 [\|(v'_{0s}, v_0)(s)\|_{L^1(\mathbb{R}^n)} + \|(v'_{0s}, v_0)(s)\|_{H^k(\mathbb{R}^n)}] ds.
\end{aligned} \tag{3.23}$$

Furthermore, if $m > \frac{n}{2}$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C \mathcal{E}_{v_0}^m t^{-\frac{n}{\alpha}} e^{-\varepsilon_1(\tilde{c}_1 - c_3)t}, \quad t > 0. \tag{3.24}$$

Particularly, when $\tilde{c}_1 = c_3$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C \mathcal{E}_{v_0}^m t^{-\frac{n}{\alpha}}, \quad t > 0. \tag{3.25}$$

Proof. Let

$$I_1(t, \eta) := (i\eta_j)^k e^{-A(\eta)(t+\tau)} e_{\tau}^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta), \tag{3.26}$$

$$I_2(t-s, \eta) := (i\eta_j)^k e^{-A(\eta)(t-s)} e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)} \left[\partial_s \hat{v}_0(s, \eta) + A(\eta) \hat{v}_0(s, \eta) \right]. \tag{3.27}$$

Then, (3.20) is reduced to

$$\partial_{x_j}^k v(t, x) = \mathcal{F}^{-1}[I_1](t, x) + \int_{-\tau}^0 \mathcal{F}^{-1}[I_2](t-s, x) ds. \tag{3.28}$$

So, by using Parseval's equality, we have

$$\begin{aligned}\|\partial_{x_j}^k v(t)\|_{L^2(\mathbb{R}^n)} &\leq \|\mathcal{F}^{-1}[I_1](t)\|_{L^2(\mathbb{R}^n)} + \int_{-\tau}^0 \|\mathcal{F}^{-1}[I_2](t-s)\|_{L^2(\mathbb{R}^n)} ds \\ &= \|I_1(t)\|_{L^2(\mathbb{R}^n)} + \int_{-\tau}^0 \|I_2(t-s)\|_{L^2(\mathbb{R}^n)} ds.\end{aligned}\quad (3.29)$$

$$\begin{aligned}|e^{-A(\eta)t}| &= e^{-c_1 t} \left| \exp \left(t \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} e^{-i y \cdot \eta} dy \right) \right| \\ &= e^{-c_1 t} \exp \left(t \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} \cos(y \cdot \eta) dy \right) \\ &= e^{-\tilde{c}_1 t} \exp \left(-t \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} (1 - \cos(y \cdot \eta)) dy \right) \\ &=: e^{-k_1 t}, \quad \text{with } k_1 := \tilde{c}_1 + \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} (1 - \cos(y \cdot \eta)) dy,\end{aligned}\quad (3.30)$$

Note that, using (3.15), (3.16), and the facts $\frac{e^x + e^{-x}}{2} \geq 1$ for all $x \in \mathbb{R}$, and $\int_{\mathbb{R}^n} J(y) \sin(y \cdot \eta) dy = 0$ because $J(y)$ is even and $\sin(y \cdot \eta)$ is odd, and $\int_{\mathbb{R}^n} J(y) dy = 1$, we have

$$\begin{aligned}&\exp \left(-t \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} (1 - \cos(y \cdot \eta)) dy \right) \\ &= \exp \left(-t \int_{\mathbb{R}^n} J(y) \frac{e^{-\lambda_* y_1} + e^{\lambda_* y_1}}{2} (1 - \cos(y \cdot \eta)) dy \right) \\ &\leq \exp \left(-t \int_{\mathbb{R}^n} J(y) (1 - \cos(y \cdot \eta)) dy \right) \\ &= \exp \left(-t \int_{\mathbb{R}^n} J(y) [1 - [\cos(y \cdot \eta) + i \sin(y \cdot \eta)]] dy \right) \\ &= e^{(\hat{J}(\eta) - 1)t}\end{aligned}\quad (3.31)$$

and

$$|B(\eta)| \leq c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1 + c\tau)} dy = c_3 =: k_2, \quad (3.32)$$

and

$$|\mathcal{B}(\eta)| = |B(\eta) e^{A(\eta)\tau}| \leq c_3 e^{k_1 \tau} = k_2 e^{k_1 \tau} =: \bar{k}_2, \quad (3.33)$$

and further

$$|e_{\tau}^{\mathcal{B}(\eta)t}| \leq e_{\tau}^{\bar{k}_2 t}. \quad (3.34)$$

If $\tilde{c}_1 \geq c_3$, from (J₂), namely, $1 - \hat{J}(\eta) = \mathcal{K}|\eta|^\alpha - o(|\eta|^\alpha) > 0$ as $\eta \rightarrow 0$, then $k_1 = \tilde{c}_1 + 1 - \hat{J}(\eta) \geq c_3 = k_2$. Using (3.30), (3.31), (3.34) and (3.9) in Lemma 3.2, we obtain

$$\begin{aligned}\|I_1(t)\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} |e^{-A(\eta)(t+\tau)} e_{\tau}^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta \\ &\leq C \int_{\mathbb{R}^n} (e^{-k_1(t+\tau)} e_{\tau}^{\bar{k}_2 t})^2 |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta \\ &\leq C \int_{\mathbb{R}^n} (e^{-\varepsilon_1(k_1 - k_2)t})^2 |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta \\ &= C e^{-2\varepsilon_1(\tilde{c}_1 - c_3)t} \int_{\mathbb{R}^n} e^{-2\varepsilon_1(1 - \hat{J}(\eta))t} |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta.\end{aligned}\quad (3.35)$$

Again from (J₂), there exist some numbers $0 < \mathcal{K}_1 < \mathcal{K}$, $0 < \delta < 1$ and $\tilde{a} > 0$, such that

$$\begin{cases} \mathcal{K}_1 |\eta|^\alpha \leq 1 - \hat{J}(\eta) \leq \mathcal{K} |\eta|^\alpha, & \text{as } |\eta| \leq \tilde{a}, \\ \delta := \mathcal{K}_1 \tilde{a}^\alpha \leq 1 - \hat{J}(\eta) \leq \mathcal{K} |\eta|^\alpha, & \text{as } |\eta| \geq \tilde{a}. \end{cases} \quad (3.36)$$

Therefore, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-2\varepsilon_1(1-\hat{J}(\eta))t} |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta \\ &= \int_{|\eta| \leq \tilde{a}} e^{-2\varepsilon_1(1-\hat{J}(\eta))t} |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta + \int_{|\eta| \geq \tilde{a}} e^{-2\varepsilon_1(1-\hat{J}(\eta))t} |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta \\ &\leq \int_{|\eta| \leq \tilde{a}} e^{-2\varepsilon_1 \mathcal{K}_1 |\eta|^\alpha t} |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta + \int_{|\eta| \geq \tilde{a}} e^{-2\varepsilon_1 \delta t} |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta \\ &\leq \|\hat{v}_0(-\tau)\|_{L^\infty(\mathbb{R}^n)}^2 t^{-\frac{n+2k}{\alpha}} \int_{|\eta| \leq \tilde{a}} e^{-2\varepsilon_1 \mathcal{K}_1 |\eta t^{\frac{1}{\alpha}}|^\alpha} |\eta_j t^{\frac{1}{\alpha}}|^{2k} d(\eta t^{\frac{1}{\alpha}}) \\ &\quad + e^{-2\varepsilon_1 \delta t} \int_{|\eta| \geq \tilde{a}} |\hat{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta \\ &\leq C(\|v_0(-\tau)\|_{L^1(\mathbb{R}^n)}^2 + \|v_0(-\tau)\|_{H^k(\mathbb{R}^n)}^2) t^{-\frac{n+2k}{\alpha}}. \end{aligned} \quad (3.37)$$

Substitute (3.37) into (3.35), we obtain

$$\|I_1(t)\|_{L^2(\mathbb{R}^n)} \leq C(\|v_0(-\tau)\|_{L^1(\mathbb{R}^n)} + \|v_0(-\tau)\|_{H^k(\mathbb{R}^n)}) t^{-\frac{n+2k}{2\alpha}} e^{-\varepsilon_1(\tilde{c}_1 - c_3)t} \quad (3.38)$$

Thus, in a similar way, we can also prove

$$\begin{aligned} & \|I_2(t-s)\|_{L^2(\mathbb{R}^n)} \\ &= \left(\int_{\mathbb{R}^n} |e^{-A(\eta)(t-s)} e^{\mathcal{B}(\eta)(t-\tau-s)}|^2 \left| \partial_s \hat{v}_0(s, \eta) + A(\eta) \hat{v}_0(s, \eta) \right|^2 \cdot |\eta_j|^{2k} d\eta \right)^{\frac{1}{2}} \\ &\leq C e^{-\varepsilon_1(\tilde{c}_1 - c_3)t} \left(\int_{\mathbb{R}^n} e^{-2\varepsilon_1(1-\hat{J}(\eta))t} (|\eta|^{2k} |\partial_s \hat{v}_0(s, \eta)| + |\eta|^{2k} |\hat{v}_0(s, \eta)|^2) d\eta \right)^{\frac{1}{2}} \\ &\leq C t^{-\frac{n+2k}{2\alpha}} e^{-\varepsilon_1(\tilde{c}_1 - c_3)t} \left(\|(\partial_s v_0, v_0)(s)\|_{L^1(\mathbb{R}^n)} + \|(\partial_s v_0, v_0)(s)\|_{H^k(\mathbb{R}^n)} \right). \end{aligned} \quad (3.39)$$

Substituting (3.38) and (3.39) to (3.29), we immediately obtain (3.22).

Similarly, we can prove (3.24). We omit the details. Thus, we complete the proof of Proposition 3.3. \square

For $\tau = 0$, the equation (3.11) is reduced to

$$\begin{cases} \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x_1} - \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} v(t, x-y) dy + c_1 v \\ \quad = c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1 + c\tau)} v(t, x-y - c\tau \mathbf{e}_1) dy, \\ v(s, x) = v_0(x), \quad x \in \mathbb{R}^n. \end{cases} \quad (3.40)$$

Taking Fourier transform to (3.40), as showed in (3.14), we have

$$\frac{d\hat{v}}{dt} = [B(\eta) - A(\eta)]\hat{v}, \quad \text{with } \hat{v}(0, \eta) = \hat{v}_0(\eta), \quad (3.41)$$

where $A(\eta)$ and $B(\eta)$ are given in (3.15) and (3.16) with $\tau = 0$, respectively. Integrating (3.41) yields

$$\hat{v}(t, \eta) = e^{-[A(\eta)-B(\eta)]t} \hat{v}_0(\eta).$$

Taking the inverse Fourier transform, we get the solution formula

$$v(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} e^{-[A(\eta)-B(\eta)]t} \hat{v}_0(\eta) d\eta.$$

Then, a similar analysis as showed before can derive the optimal decay of the solution in the case without time-delay as follows. The detail of proof is omitted.

Proposition 3.4 (Optimal decay rates for $\tau = 0$) *Suppose that $v_0 \in H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for $m \geq 0$, then the solution of the linearized equation (3.40) satisfies*

$$\|\partial_{x_j}^k v(t)\|_{L^2(\mathbb{R}^n)} \leq C(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)}) t^{-\frac{n+2k}{2\alpha}} e^{-(\tilde{c}_1 - c_3)t}, \quad t > 0, \quad (3.42)$$

for $k = 0, 1, \dots, [m]$ and $j = 1, \dots, n$, where the positive constants \tilde{c}_1 and c_3 are defined in (3.21) for $\tau = 0$.

Furthermore, if $m > \frac{n}{2}$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)}) t^{-\frac{n}{\alpha}} e^{-(\tilde{c}_1 - c_3)t}, \quad t > 0. \quad (3.43)$$

Particularly, when $\tilde{c}_1 = c_3$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)}) t^{-\frac{n}{\alpha}}, \quad t > 0. \quad (3.44)$$

4 Global Existence and Comparison principle

In this section, we prove the global existence and uniqueness of the solution for the Cauchy problem to the nonlinear equation with nonlocal dispersion (1.7), and then establish the comparison principle in n -D case by a different proof approach to the previous work [5, 12].

Proposition 4.1 (Existence and Uniqueness) *Let $u_0(s, x) \in C([- \tau, 0]; C(\mathbb{R}^n))$ with $0 = u_- \leq u_0(s, x) \leq u_+$ for $(s, x) \in [- \tau, 0] \times \mathbb{R}^n$, then the solution to (1.7) uniquely and globally exists, and satisfies that $u \in C^1([0, \infty); C(\mathbb{R}^n))$, and $u_- \leq u(t, x) \leq u_+$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.*

Proof. Multiplying (1.7) by $e^{\eta_0 t}$ and integrating it over $[0, t]$ with respect to t , where $\eta_0 > 0$ will be technically selected in (4.4) below, we then express (1.7) in the integral form

$$\begin{aligned} u(t, x) = & e^{-\eta_0 t} u(0, x) + \int_0^t e^{-\eta_0(t-s)} \left[\int_{\mathbb{R}^n} J(x-y) u(s, y) dy + (\eta_0 - 1) u(s, x) \right. \\ & \left. - d(u(s, x)) + \int_{\mathbb{R}^n} f_\beta(y) b(u(s-\tau, x-y)) dy \right] ds. \end{aligned} \quad (4.1)$$

Let us define the solution space as, for any $T \in [0, \infty]$,

$$\begin{aligned} \mathfrak{B} = & \left\{ u(t, x) \mid u(t, x) \in C([0, T] \times \mathbb{R}^n) \text{ with } u_- \leq u \leq u_+, \right. \\ & \left. u(s, x) = u_0(s, x), (s, x) \in [- \tau, 0] \times \mathbb{R}^n \right\}, \end{aligned} \quad (4.2)$$

with the norm

$$\|u\|_{\mathfrak{B}} = \sup_{t \in [0, T]} e^{-\eta_0 t} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \quad (4.3)$$

where

$$\eta_0 := 1 + \eta_1 + \eta_2, \quad \eta_1 := \max_{u \in [u_-, u_+]} |d'(u)|, \quad \eta_2 := \max_{u \in [u_-, u_+]} |b'(u)|. \quad (4.4)$$

Clearly, \mathfrak{B} is a Banach space.

Define an operator \mathcal{P} on \mathfrak{B} by

$$\begin{aligned} \mathcal{P}(u)(t, x) := & e^{-\eta_0 t} u_0(0, x) + \int_0^t e^{-\eta_0(t-s)} \left[\int_{\mathbb{R}^n} J(x-y) u(s, y) dy + (\eta_0 - 1) u(s, x) \right. \\ & \left. - d(u(s, x)) + \int_{\mathbb{R}^n} f_\beta(y) b(u(s-\tau, x-y)) dy \right] ds, \quad \text{for } 0 \leq t \leq T, \end{aligned} \quad (4.5)$$

and

$$\mathcal{P}(u)(s, x) := u_0(s, x), \quad \text{for } s \in [-\tau, 0]. \quad (4.6)$$

Now we are going to prove that \mathcal{P} is a contracting operator from \mathfrak{B} to \mathfrak{B} .

Firstly, we prove that $\mathcal{P} : \mathfrak{B} \rightarrow \mathfrak{B}$. In fact, if $u \in \mathfrak{B}$, from (H₁)-(H₃), namely, $0 = d(0) \leq d(u) \leq d(u_+)$, $0 = b(0) \leq b(u) \leq b(u_+)$, and $d(u_+) = b(u_+)$, and using the facts $\int_{\mathbb{R}^n} J(x-y) dy = 1$, $\int_{\mathbb{R}^n} f_\beta(y) dy = 1$, and

$$g(u) := (\eta_0 - 1)u - d(u) \text{ is increasing}, \quad (4.7)$$

which implies $g(u_+) \geq g(u) \geq g(0) = 0$ for $u \in [u_-, u_+]$, then we have

$$\begin{aligned} 0 = u_- \leq \mathcal{P}(u) & \leq e^{-\eta_0 t} u_+ + \int_0^t e^{-\eta_0(t-s)} \left[\int_{\mathbb{R}^n} J(x-y) u_+ dy \right. \\ & \quad \left. + (\eta_0 - 1) u_+ - d(u_+) + \int_{\mathbb{R}^n} f_\beta(y) b(u_+) dy \right] ds \\ & = e^{-\eta_0 t} u_+ + \int_0^t e^{-\eta_0(t-s)} [\eta_0 u_+ - d(u_+) + b(u_+)] ds \\ & = u_+. \end{aligned} \quad (4.8)$$

This plus the continuity of $\mathcal{P}(u)$ based on the continuity of u proves $\mathcal{P}(u) \in \mathfrak{B}$, namely, \mathcal{P} maps from \mathfrak{B} to \mathfrak{B} .

Secondly, we prove that \mathcal{P} is contracting. In fact, let $u_1, u_2 \in \mathfrak{B}$, and $v = u_1 - u_2$, then we have

$$\mathcal{P}(u_1) - \mathcal{P}(u_2) \quad (4.9)$$

$$\begin{aligned} & = \int_0^t e^{-\eta_0(t-s)} \left[\int_{\mathbb{R}^n} J(x-y) v(s, y) dy + (\eta_0 - 1) v(s, x) - [d(u_1(s, x)) - d(u_2(s, x))] \right. \\ & \quad \left. + \int_{\mathbb{R}^n} f_\beta(y) [b(u_1(s-\tau, x-y)) - b(u_2(s-\tau, x-y))] dy \right] ds. \end{aligned} \quad (4.10)$$

So, we have

$$\begin{aligned} |\mathcal{P}(u_1) - \mathcal{P}(u_2)| e^{-\eta_0 t} & \leq \int_0^t e^{-2\eta_0(t-s)} \left(\eta_0 + \max_{u \in [u_-, u_+]} |d'(u)| \right) \|v\|_{\mathfrak{B}} ds \\ & \quad + \max_{u \in [u_-, u_+]} |b'(u)| \begin{cases} \int_0^{t-\tau} e^{-2\eta_0(t-s)} \|v\|_{\mathfrak{B}} ds, & \text{for } t \geq \tau \\ 0, & \text{for } 0 \leq t \leq \tau \end{cases} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\eta_0} \left((\eta_0 + \eta_1)(1 - e^{-2\eta_0 t}) + \eta_2(e^{-2\eta_0 \tau} - e^{-2\eta_0 t}) \right) \|v\|_{\mathfrak{B}} \\
&\leq \frac{\eta_0 + \eta_1 + \eta_2}{2\eta_0} \|v\|_{\mathfrak{B}} \\
&= \frac{2\eta_0 - 1}{2\eta_0} \|v\|_{\mathfrak{B}} \\
&=: \rho \|v\|_{\mathfrak{B}}
\end{aligned} \tag{4.11}$$

for $0 < \rho := \frac{2\eta_0 - 1}{2\eta_0} < 1$, namely, we prove that the mapping \mathcal{P} is contracting:

$$\|\mathcal{P}(u_1) - \mathcal{P}(u_2)\|_{\mathfrak{B}} \leq \rho \|u_1 - u_2\|_{\mathfrak{B}} < \|u_1 - u_2\|_{\mathfrak{B}}. \tag{4.12}$$

Hence, by the Banach fixed-point theorem, \mathcal{P} has a unique fixed point u in \mathfrak{B} , i.e, the integral equation (4.1) has a unique classical solution on $[0, T]$ for any given $T > 0$. Differentiating (4.1) with respect to t , we get back to the original equation (1.7), i.e.,

$$u_t = J * u - u + d(u(t, x)) + \int_{\mathbb{R}^n} f_{\beta}(y) b(u(t - \tau, x - y)) dy, \tag{4.13}$$

then we can easily confirm from the right-hand-side of (4.13) that $u_t \in C([0, T] \times \mathbb{R}^n)$. This completes our proof. \square

Remark 4.2 From the proof of Proposition 4.1, we realize that, when $u_0(s, x) \in C^k([-\tau, 0] \times \mathbb{R}^n)$, then the solution of the time-delayed equation (1.7) holds $u(t, x) \in C^{k+1}([0, \infty); C(\mathbb{R}^n))$; while for the non-delayed equation (1.8) (i.e., $\tau = 0$), if $u_0(x) \in C(\mathbb{R}^n)$, then the solution of the non-delayed equation (1.8) holds $u(t, x) \in C^\infty([0, \infty); C(\mathbb{R}^n))$. This means that the solution to the nonlocal dispersion equation (1.7) possesses a really good regularity in time. However, the solutions for (1.7) lack the regularity in space.

Now we establish two comparison principle for (1.7). Although the comparison principle in 1D case were proved in [5, 12]. Here we give a comparison principle in n -D case with much weaker restriction on the initial data. The proof is also new and easy to follow. Different from the previous works [5, 12], instead of the differential equation (1.7), we will work on the integral equation (4.1), and sufficiently use the property of contracting operator \mathcal{P} .

Let $\bar{u}(t, x)$ be an upper solution to (1.7), namely

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} - J * \bar{u} + \bar{u} + d(\bar{u}(t, x)) \geq \int_{\mathbb{R}^n} f_{\beta}(y) b(\bar{u}(t - \tau, x - y)) dy, \\ \bar{u}(s, x) \geq u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n, \end{cases} \tag{4.14}$$

where its integral form can be written as

$$\begin{aligned} \bar{u}(t, x) \geq & e^{-\eta_0 t} \bar{u}(0, x) + \int_0^t e^{-\eta_0(t-s)} \left[\int_{\mathbb{R}^n} J(x - y) \bar{u}(s, y) dy + (\eta_0 - 1) \bar{u}(s, x) \right. \\ & \left. - d(\bar{u}(s, x)) + \int_{\mathbb{R}^n} f_{\beta}(y) b(\bar{u}(s - \tau, x - y)) dy \right] ds, \quad \text{for } t > 0 \end{aligned} \tag{4.15}$$

and let $\underline{u}(t, x)$ be a lower solution to (1.7) satisfying (4.14) or (4.15) conversely. Then we have the following comparison result.

Proposition 4.3 (Comparison Principle) *Let $\underline{u}(t, x)$ and $\bar{u}(t, x)$ be the classical lower and upper solutions to (1.7), with $u_- \leq \underline{u}(t, x)$, $\bar{u}(t, x) \leq u_+$, respectively, and satisfy $0 \leq \underline{u}(t, x) \leq u_+$ and $0 \leq \bar{u}(t, x) \leq u_+$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then $\underline{u}(t, x) \leq \bar{u}(t, x)$ for $(t, x) \in [0, \infty) \times \mathbb{R}^n$.*

Proof. We need to prove $\bar{u}(t, x) - \underline{u}(t, x) \geq 0$ for $(t, x) \in [0, \infty) \times \mathbb{R}^n$, namely, $r(t) := \inf_{x \in \mathbb{R}^n} v(t, x) \geq 0$, where $v(t, x) := \bar{u}(t, x) - \underline{u}(t, x)$.

If this is not true, then there exist some constants $\varepsilon > 0$ and $T > 0$ such that $r(t) > -\varepsilon e^{3\eta_0 t}$ for $t \in [0, T]$ and $r(T) = -\varepsilon e^{3\eta_0 T}$, where η_0 given in (4.4).

Since $\underline{u}(t, x)$ and $\bar{u}(t, x)$ are the lower and upper solutions to (1.7) and $\bar{u}(s, x) - \underline{u}(s, x) \geq 0$, for $s \in [-\tau, 0]$, and using (4.4) and (4.7), and noting $\bar{u}(t, x) - \underline{u}(t, x) \geq -\varepsilon e^{3\eta_0 T}$ for $(t, x) \in [0, T] \times \mathbb{R}^n$, then we have, for $0 \leq t \leq T$,

$$\begin{aligned}
& \bar{u}(t, x) - \underline{u}(t, x) \\
& \geq e^{-\eta_0 t} [\bar{u}(0, x) - \underline{u}(0, x)] \\
& \quad + \int_0^t e^{-\eta_0(t-s)} \left(\int_{\mathbb{R}^n} J(x-y) [\bar{u}(s, y) - \underline{u}(s, y)] dy \right. \\
& \quad \left. + g(\bar{u}(s, x)) - g(\underline{u}(s, x)) \right. \\
& \quad \left. + \int_{\mathbb{R}^n} f_\beta(y) [b(\bar{u}(s-\tau, x-y)) - b(\underline{u}(s-\tau, x-y))] dy \right) ds \\
& \geq \int_0^t e^{-\eta_0(t-s)} \left(-\varepsilon e^{3\eta_0 s} - \max_{\zeta \in [u_-, u_+]} g'(\zeta) \varepsilon e^{3\eta_0 s} \right) ds \\
& \quad - \max_{u \in [u_-, u_+]} |b'(u)| \begin{cases} \int_\tau^t e^{-\eta_0(t-s)} \varepsilon e^{3\eta_0(s-\tau)} ds, & \text{for } t \geq \tau \\ 0, & \text{for } 0 \leq t \leq \tau \end{cases} \\
& \geq \begin{cases} -(\eta_0 + 1) \varepsilon e^{-\eta_0 t} \int_0^t e^{4\eta_0 s} ds - \eta_0 \varepsilon e^{-3\eta_0 \tau} e^{-\eta_0 t} \int_\tau^t e^{4\eta_0 s} ds, & \text{for } t \geq \tau \\ -(\eta_0 + 1) \varepsilon e^{-\eta_0 t} \int_0^t e^{4\eta_0 s} ds, & \text{for } 0 \leq t \leq \tau \end{cases} \\
& \geq -\frac{2\eta_0 + 1}{4\eta_0} \varepsilon e^{3\eta_0 t}. \tag{4.16}
\end{aligned}$$

Thus, from the assumption we know

$$-\varepsilon e^{3\eta_0 T} = \inf_{x \in \mathbb{R}^n} (\bar{u}(T, x) - \underline{u}(T, x)) \geq -\frac{2\eta_0 + 1}{4\eta_0} \varepsilon e^{3\eta_0 T}, \tag{4.17}$$

which is a contradiction for $\eta_0 > \frac{1}{2}$. Here, our η_0 defined in (4.4) satisfies $\eta_0 > 1$. Thus the proof is complete. \square

5 Global Stability of Planar Traveling Waves

The main purpose in this section is to prove Theorems 2.2 for all traveling waves including the critical traveling waves.

For given traveling wave $\phi(x_1 + ct)$ with the speed $c \geq c_*$ and the given initial data $u_- \leq u_0(s, x) \leq u_+$ for $(s, x) \in [-\tau, 0] \times \mathbb{R}^n$, let us define $U_0^+(s, x)$ and $U_0^-(s, x)$ as

$$\begin{aligned}
U_0^-(s, x) &:= \min\{\phi(x_1 + cs), u_0(s, x)\} \\
U_0^+(s, x) &:= \max\{\phi(x_1 + cs), u_0(s, x)\} \tag{5.1}
\end{aligned}$$

for $(s, x) \in [-\tau, 0] \times \mathbb{R}^n$. So,

$$u_0 - \phi = (U_0^+ - \phi) + (U_0^- - \phi).$$

Since $u_0 - \phi \in C([-\tau, 0]; H_w^{m+1}(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ with $m > \frac{n}{2}$ and $w(x) \geq 1$ (see (2.11)), and noting Sobolev's embedding theorem $H^m(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$, we have $u_0 - \phi \in C([-\tau, 0]; C(\mathbb{R}^n))$. On the other hand, the traveling wave $\phi(x_1 + cs)$ is smooth, then we can guarantee $U_0^\pm(s, x) \in C([-\tau, 0]; C(\mathbb{R}^n))$. Thus, applying Proposition 4.1, we know that the solutions of (1.7) with the initial data $U_0^+(s, x)$ and $U_0^-(s, x)$ globally exist, and denote them by $U^+(t, x)$ and $U^-(t, x)$, respectively, that is,

$$\begin{cases} \frac{\partial U^\pm}{\partial t} - J * U^\pm + U^\pm + d(U^\pm) = \int_{\mathbb{R}^n} f_\beta(y) b(U^\pm(t - \tau, x - y)) dy, \\ U^\pm(s, x) = U_0^\pm(s, x), \quad x \in \mathbb{R}^n, \quad s \in [-\tau, 0]. \end{cases} \quad (5.2)$$

Then the comparison principle (Proposition 4.3) further implies

$$\begin{cases} u_- \leq U^-(t, x) \leq u(t, x) \leq U^+(t, x) \leq u_+ \\ u_- \leq U^-(t, x) \leq \phi(x_1 + ct) \leq U^+(t, x) \leq u_+ \end{cases} \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n. \quad (5.3)$$

In what follows, we are going to complete the proof for the stability in three steps.

Step 1. The convergence of $U^+(t, x)$ to $\phi(x_1 + ct)$

Let

$$V(t, x) := U^+(t, x) - \phi(x_1 + ct), \quad V_0(s, x) := U_0^+(s, x) - \phi(x_1 + cs). \quad (5.4)$$

It follows from (5.3) that

$$V(t, x) \geq 0, \quad V_0(s, x) \geq 0. \quad (5.5)$$

We see from (1.7) that $V(t, x)$ satisfies (by linearizing it around 0)

$$\begin{aligned} & \frac{\partial V}{\partial t} - \int_{\mathbb{R}^n} J(y) V(t, x - y) dy + V + d'(0)V \\ & \quad - b'(0) \int_{\mathbb{R}^n} f_\beta(y) V(t - \tau, x - y) dy \\ & = -Q_1(t, x) + \int_{\mathbb{R}^n} f_\beta(y) Q_2(t - \tau, x - y) dy + [d'(0) - d'(\phi(x_1 + ct))]V \\ & \quad + \int_{\mathbb{R}^n} f_\beta(y) [b'(\phi(x_1 - y_1 + c(t - \tau))) - b'(0)] V(t - \tau, x - y) dy \\ & =: I_1(t, x) + I_2(t - \tau, x) + I_3(t, x) + I_4(t - \tau, x), \end{aligned} \quad (5.6)$$

with the initial data

$$V(s, x) = V_0(s, x), \quad s \in [-\tau, 0], \quad (5.7)$$

where

$$Q_1(t, x) = d(\phi + V) - d(\phi) - d'(\phi)V \quad (5.8)$$

with $\phi = \phi(x_1 + ct)$ and $V = V(t, x)$, and

$$Q_2(t - \tau, x - y) = b(\phi + V) - b(\phi) - b'(\phi)V \quad (5.9)$$

with $\phi = \phi(x_1 - y_1 + c(t - \tau))$ and $V = V(t - \tau, x - y)$. Here I_i , $i = 1, 2, 3, 4$, denotes the i -th term in the right-side of line above (5.6).

From (H_3) , i.e., $d''(u) \geq 0$ and $b''(u) \leq 0$, applying Taylor formula to (5.8) and (5.9), we immediately have

$$Q_1(t, x) \geq 0 \quad \text{and} \quad Q_2(t - \tau, x - y) \leq 0,$$

which implies

$$I_1(t, x) \leq 0 \quad \text{and} \quad I_2(t - \tau, x) \leq 0. \quad (5.10)$$

From (H_3) again, since $d'(\phi)$ is increasing and $b'(\phi)$ is decreasing, then $d'(0) - d'(\phi(x_1 + ct)) \leq 0$ and $b'(\phi(x_1 - y_1 + c(t - \tau))) - b'(0) \leq 0$, which imply, with $V \geq 0$,

$$I_3(t, x) \leq 0 \quad \text{and} \quad I_4(t - \tau, x) \leq 0. \quad (5.11)$$

Thus, applying (5.10) and (5.11) to (5.6), we obtain

$$\frac{\partial V}{\partial t} - J * V + V + d'(0)V - b'(0) \int_{\mathbb{R}^n} f_\beta(y) V(t - \tau, x - y) dy \leq 0. \quad (5.12)$$

Let $\bar{V}(t, x)$ be the solution of the following equation with the same initial data $V_0(s, x)$:

$$\begin{cases} \frac{\partial \bar{V}}{\partial t} - J * \bar{V} + \bar{V} + d'(0)\bar{V} - b'(0) \int_{\mathbb{R}^n} f_\beta(y) \bar{V}(t - \tau, x - y) dy = 0, & (t, x) \in R_+ \times \mathbb{R}^n, \\ \bar{V}(s, x) = V_0(s, x), & s \in [-\tau, 0], x \in \mathbb{R}^n. \end{cases} \quad (5.13)$$

From Proposition 4.1, we know that $\bar{V}(t, x)$ globally exists. Furthermore, (5.13) is actually a linear equation, and its solution is as smooth as its initial data. By the comparison principle (Proposition 4.3), we have

$$0 \leq V(t, x) \leq \bar{V}(t, x), \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n. \quad (5.14)$$

Let

$$v(t, x) := e^{-\lambda_*(x_1 + ct - x_*)} \bar{V}(t, x). \quad (5.15)$$

From (5.13), $v(t, x)$ satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} - \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} v(t, x - y) dy + c_1 v \\ = c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1 + c\tau)} v(t - \tau, x - y) dy, \end{aligned} \quad (5.16)$$

where

$$c_1 := c\lambda_* + 1 + d'(0) > 0, \quad \text{and} \quad c_2 := b'(0). \quad (5.17)$$

When $\tau = 0$, then (3.40) is reduced to

$$\frac{\partial v}{\partial t} - \int_{\mathbb{R}^n} J(y) e^{-\lambda_* y_1} v(t, x - y) dy + c_1 v = c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_* y_1} v(t, x - y) dy. \quad (5.18)$$

Applying Proposition 3.3 to (5.16) for $\tau > 0$ and Proposition 3.4 to (5.18) for $\tau = 0$, we obtain the following decay rates:

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{n}{\alpha}} e^{-\varepsilon_1(\tilde{c}_1 - c_3)t}, \quad \text{for } \tau > 0, \quad (5.19)$$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{n}{\alpha}} e^{-(\tilde{c}_1 - c_3)t}, \quad \text{for } \tau = 0, \quad (5.20)$$

where $0 < \varepsilon_1 = \varepsilon_1(\tau) < 1$, and c_3 is defined in (3.21), which can be directly calculated as, by using the property (1.11),

$$\begin{aligned} c_3 &= b'(0) \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_*(y_1+c\tau)} dy \\ &= b'(0) \int_{\mathbb{R}} f_{1\beta}(y_1) e^{-\lambda_*(y_1+c\tau)} dy_1 \\ &= b'(0) e^{\beta\lambda_*^2 - \lambda_*c\tau} > 0. \end{aligned} \quad (5.21)$$

and

$$\tilde{c}_1 = c\lambda_* + 1 + d'(0) - \int_{\mathbb{R}} J(y_1) e^{-\lambda_*y_1} dy_1 = c\lambda_* + d'(0) - E_c(\lambda_*). \quad (5.22)$$

When $c > c_*$, namely, the wave $\phi(x_1 + ct)$ is non-critical, from (2.7) in Theorem 2.1, we realize

$$\tilde{c}_1 := c\lambda_* + d'(0) - E_c(\lambda_*) = G_c(\lambda_*) > H_c(\lambda_*) = b'(0) e^{\beta\lambda_*^2 - \lambda_*c\tau} =: c_3. \quad (5.23)$$

Thus, (5.19) and (5.20) immediately imply the following exponential decay for $c > c_*$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{\alpha}} e^{-\varepsilon_1 \tilde{\mu} t}, \quad \text{for } \tau > 0, \quad (5.24)$$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{\alpha}} e^{-\tilde{\mu} t}, \quad \text{for } \tau = 0, \quad (5.25)$$

where

$$\tilde{\mu} := \tilde{c}_1 - c_3 = G_c(\lambda_*) - H_c(\lambda_*) > 0. \quad (5.26)$$

When $c = c_*$, namely, the wave $\phi(x_1 + c_*t)$ is critical, from (2.8) in Proposition 2.1, we realize

$$\tilde{c}_1 := c\lambda_* + d'(0) - E_c(\lambda_*) = G_c(\lambda_*) = H_c(\lambda_*) = b'(0) e^{\beta\lambda_*^2 - \lambda_*c\tau} := c_3. \quad (5.27)$$

Then, from (5.19) and (5.20), we immediately obtain the following algebraic decay for $c = c_*$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \geq 0. \quad (5.28)$$

Since $V(t, x) \leq \bar{V}(t, x) = e^{\lambda_*(x_1+ct-x_*)} v(t, \xi)$, and $0 < e^{\lambda_*(x_1+ct-x_*)} \leq 1$ for $x_1 \in (-\infty, x_* - ct]$, we immediately obtain the following decay for V .

Lemma 5.1 *Let $V = V(t, x)$. Then*

1. *when $c > c_*$, then*

$$\|V(t)\|_{L^\infty((-\infty, x_*-ct] \times \mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\varepsilon_1 \tilde{\mu} t}, \quad \text{for } \tau > 0, \quad (5.29)$$

$$\|V(t)\|_{L^\infty((-\infty, x_*-ct] \times \mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\tilde{\mu} t}, \quad \text{for } \tau = 0; \quad (5.30)$$

Here $\tilde{\mu} := \tilde{c}_1 - c_3 = G_c(\lambda_*) - H_c(\lambda_*) > 0$ for $c > c_*$.

2. *when $c = c_*$, then*

$$\|V(t)\|_{L^\infty((-\infty, x_*-ct] \times \mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \geq 0. \quad (5.31)$$

Next we prove $V(t, x)$ exponentially decay for $x \in [x_* - ct, \infty) \times \mathbb{R}^{n-1}$.

Lemma 5.2 For $\tau > 0$, it holds that

$$\|V(t)\|_{L^\infty([x_*-ct, \infty) \times \mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } c > c_*, \quad (5.32)$$

$$\|V(t)\|_{L^\infty([x_*-ct, \infty) \times \mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}}, \quad \text{for } c = c_*, \quad (5.33)$$

with some constant $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1 \tilde{\mu}\}$ for $c > c_*$.

Proof. From (5.2) and (1.9), as set in (5.4) $V(t, x) := U^+(t, x) - \phi(x_1 + ct)$, we have

$$\frac{\partial V}{\partial t} - J * V + V + d(\phi + V) - d(\phi) = \int_{\mathbb{R}^n} f_\beta(y) [b(\phi + V) - b(\phi)] dy. \quad (5.34)$$

Applying Taylor expansion formula and noting (H₃) for $d''(u) \geq 0$ and $b''(u) \leq 0$, we have

$$d(\phi + V) - d(\phi) = d'(\phi)V + d''(\bar{\phi}_1)V^2 \geq d'(\phi)V, \quad (5.35)$$

$$b(\phi + V) - b(\phi) = b'(\phi)V + b''(\bar{\phi}_2)V^2 \leq b'(\phi)V, \quad (5.36)$$

where $\bar{\phi}_i$ ($i = 1, 2$) are some functions between ϕ and $\phi + V$. Substituting (5.35) and (5.36) into (5.34), and noticing Lemma 5.1, we have

$$\begin{cases} \frac{\partial V}{\partial t} - J * V + V + d'(\phi)V \leq \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(x_1 - y_1 + c(t - \tau))) V(t - \tau, x - y) dy, & \text{for } t > 0, x \in \mathbb{R}^n \\ V|_{x_1 \leq x_* - ct} \leq C_2(1 + t)^{-\frac{n}{\alpha}} e^{-\varepsilon_1 \tilde{\mu} t}, & \text{for } t > 0, (x_2, \dots, x_n) \in \mathbb{R}^{n-1} \\ V|_{t=s} = V_0(s, x), & \text{for } s \in [-\tau, 0], x \in \mathbb{R}^n \end{cases} \quad (5.37)$$

for some positive constant C_2 .

Let

$$\tilde{V}(t) = C_3(1 + \tau + t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \quad (5.38)$$

for $C_3 \geq C_2 \geq \max_{(s,x) \in [-\tau, 0] \times \mathbb{R}^n} |V_0(s, x)|$. As in (2.12), for given $0 < \varepsilon_0 < 1$, we can select a sufficiently large number x_* such that, for $\xi_1 \geq x_* \gg 1$,

$$d'(\phi(\xi_1)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy \geq \varepsilon_0 [d'(u_+) - b'(u_+)] > 0. \quad (5.39)$$

Thus, we have

$$\begin{aligned} & \frac{\partial \tilde{V}}{\partial t} - J * \tilde{V} + \tilde{V} + d'(\phi)\tilde{V} - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) \tilde{V}(t - \tau) dy \\ &= -\frac{n}{\alpha} C_3(1 + t + \tau)^{-\frac{n}{\alpha}-1} e^{-\mu_\tau t} - \mu_\tau C_3(1 + t + \tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \\ & \quad + C_3(1 + t + \tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} d'(\phi(\xi_1)) \\ & \quad - C_3(1 + t)^{-\frac{n}{\alpha}} e^{-\mu_\tau(t-\tau)} \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy \\ &= C_3(1 + t + \tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \left\{ \left[d'(\phi(\xi_1)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy \right] - \mu_\tau \right. \\ & \quad \left. - \frac{n}{\alpha} (1 + t + \tau)^{-1} - \left(e^{\mu_\tau \tau} \left(\frac{1 + t}{1 + t + \tau} \right)^{-\frac{n}{\alpha}} - 1 \right) \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy \right\} \\ &\geq C_3(1 + t + \tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \left\{ \varepsilon_0 [d'(u_+) - b'(u_+)] - \mu_\tau - \frac{n}{\alpha} (1 + t + \tau)^{-1} \right\} \end{aligned}$$

$$\begin{aligned}
& - \left(e^{\mu_\tau \tau} \left(\frac{1+t}{1+t+\tau} \right)^{-\frac{n}{\alpha}} - 1 \right) \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy \Big\} \\
& \geq 0
\end{aligned} \tag{5.40}$$

by selecting a sufficiently small number

$$0 < \mu_\tau < d'(u_+) - b'(u_+) \quad \text{for } c > c_*, \tag{5.41}$$

$$\mu_\tau = 0 \quad \text{for } c = c_*, \tag{5.42}$$

and taking $t \geq l_0 \tau$ for a sufficiently large integer $l_0 \gg 1$. Hence, we proved that

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - J * \tilde{V} + \tilde{V} + d'(\phi) \tilde{V} \geq \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) \tilde{V}(t - \tau) dy, \\ \quad \text{for } t > l_0 \tau, \xi \in [x_*, +\infty) \times \mathbb{R}^{n-1} \\ \tilde{V}|_{\xi_1=x_*} = C_3(1+\tau+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} > C_2(1+t)^{-\frac{n}{\alpha}} e^{-\varepsilon_1 \tilde{\mu} t}, \quad \text{for } t > 0, (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1} \\ \tilde{V}(t) = C_3(1+\tau+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} > V_0(t, \xi), \quad \text{for } t \in [-\tau, l_0 \tau], \xi \in \mathbb{R}^n. \end{cases} \tag{5.43}$$

Denote $\Omega := \{(x, t) | x_1 \geq x_* - ct, t \geq l_0 \tau, (x_2, \dots, x_n) \in \mathbb{R}^{n-1}\}$. Noticing the construction of (5.37) and (5.43), then similar to the proof of Proposition 4.3, we know that

$$\tilde{V}(t) - V(t, x) \geq 0, \quad \text{for } (x, t) \in \mathbb{R}^n \times [-\tau, \infty) \setminus \Omega. \tag{5.44}$$

Thus the proof is complete. \square

For $\tau = 0$, it is easy to prove the corresponding results as follows.

Lemma 5.3 *For $\tau = 0$, it holds that*

$$\|V(t)\|_{L^\infty([x_*-ct, \infty) \times \mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } c > c_*, \tag{5.45}$$

$$\|V(t)\|_{L^\infty([x_*-ct, \infty) \times \mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}}, \quad \text{for } c = c_*, \tag{5.46}$$

with some constant $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1 \tilde{\mu}\}$ for $c > c_*$.

Combing Lemma 5.1-Lemma 5.3, we obtain the decay rates for $V(t, x)$ in $L^\infty(\mathbb{R}^n)$.

Lemma 5.4 *It holds that:*

1. *when $c > c_*$, then*

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } \tau > 0, \tag{5.47}$$

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \tag{5.48}$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1 [G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

2. *when $c = c_*$,*

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \geq 0. \tag{5.49}$$

Since $V(t, x) = U^+(t, x) - \phi(x_1 + ct)$, Lemma 5.4 give directly the following convergence for the solution in the cases with time-delay.

Lemma 5.5 *It holds that:*

1. *when $c > c_*$, then*

$$\sup_{x \in \mathbb{R}^n} |U^+(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } \tau > 0, \quad (5.50)$$

$$\sup_{x \in \mathbb{R}^n} |U^+(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \quad (5.51)$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

2. *when $c = c_*$, then*

$$\sup_{x \in \mathbb{R}^n} |U^+(t, x) - \phi(x_1 + c_*t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \geq 0. \quad (5.52)$$

Step 2. The convergence of $U^-(t, x)$ to $\phi(x_1 + ct)$

For the traveling wave $\phi(x_1 + ct)$ with $c \geq c_*$, let

$$V(t, x) = \phi(x_1 + ct) - U^-(t, x), \quad V_0(s, x) = \phi(x_1 + cs) - U_0^-(s, x). \quad (5.53)$$

As in Step 1, we can similarly prove that $U^-(t, x)$ converges to $\phi(x_1 + ct)$ as follows.

Lemma 5.6 *It holds that:*

1. *when $c > c_*$, then*

$$\sup_{x \in \mathbb{R}^n} |U^-(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } \tau > 0, \quad (5.54)$$

$$\sup_{x \in \mathbb{R}^n} |U^-(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \quad (5.55)$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

2. *when $c = c_*$, then*

$$\sup_{x \in \mathbb{R}^n} |U^-(t, x) - \phi(x_1 + c_*t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \geq 0. \quad (5.56)$$

Step 3. The convergence of $u(t, x)$ to $\phi(x_1 + ct)$

Finally, we prove that $u(t, x)$ converges to $\phi(x_1 + ct)$. Since the initial data satisfy $U_0^-(s, x) \leq u_0(s, x) \leq U_0^+(s, x)$ for $(s, x) \in [-\tau, 0] \times \mathbb{R}^n$, then the comparison principle implies that

$$U^-(t, x) \leq u(t, x) \leq U^+(t, x), \quad (t, x) \in R_+ \times \mathbb{R}^n.$$

Thanks to Lemmas 5.5 and 5.6, by the squeeze argument, we have the following convergence results.

Lemma 5.7 *It holds that:*

1. when $c > c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } \tau > 0, \quad (5.57)$$

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \quad (5.58)$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

2. when $c = c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_*t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \geq 0. \quad (5.59)$$

6 Applications and Concluding Remark

In this section, we first give the direct applications of Theorem 2.1-2.2 to the Nicholson's blowflies type equation with nonlocal dispersion, and the classical Fisher-KPP equation with nonlocal dispersion. Then we point out that, the developed stability theory above can be also applied to the more general case.

6.1 Nicholson's blowflies equation with nonlocal dispersion

For the equation (1.7), by taking $d(u) = \delta u$ and $b(u) = pue^{-au}$ with $\delta > 0$, $p > 0$ and $a > 0$, we get the so-called Nicholson's blowflies equation with nonlocal dispersion

$$\begin{cases} \frac{\partial u}{\partial t} - J * u + u + \delta u(t, x) = p \int_{\mathbb{R}^n} f_\beta(y) u(t - \tau, x - y) e^{-au(t-\tau, x-y)} dy, \\ u(s, x) = u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n. \end{cases} \quad (6.1)$$

Clearly, there exist two constant equilibria $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{p}{\delta}$, and the selected $d(u)$ and $b(u)$ satisfy the hypothesis (H₁)-(H₃) automatically under the consideration of $1 < \frac{p}{\delta} \leq e$. Let $J(x)$ satisfy the hypothesis (J₁) and (J₂), from Theorem 2.1 and Theorem 2.2, we have the following existence of monostable traveling waves and their stabilities.

Theorem 6.1 (Traveling waves) *Let $J(x)$ satisfy (J₁) and (J₂). For (6.1), there exists the minimal speed $c_* > 0$, such that*

- when $c \geq c_*$, the planar traveling waves $\phi(x \cdot \mathbf{e}_1 + ct)$ exist uniquely (up to a shift);
- when $c < c_*$, the planar traveling waves $\phi(x \cdot \mathbf{e}_1 + ct)$ do not exist;

Here $c_* > 0$ and $\lambda_* > 0$ are determined by

$$H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \quad \text{and} \quad H'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*),$$

where

$$H_c(\lambda) = pe^{\beta\lambda^2 - \lambda c\tau} \quad \text{and} \quad G_c(\lambda) = c\lambda - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda y_1} dy_1 + 1 + \delta.$$

Particularly, when $c > c_*$, then $H_c(\lambda_*) < G_c(\lambda_*)$.

Theorem 6.2 (Stability of traveling waves) *Let $J(x)$ satisfy (J_1) and (J_2) , and the initial data be $u_0 - \phi \in C([- \tau, 0]; H_w^m(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ and $\partial_s(u_0 - \phi) \in L^1([- \tau, 0]; H_w^{m+1}(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ with $m > \frac{n}{2}$, and $u_- \leq u_0 \leq u_+$ for $(s, x) \in [- \tau, 0] \times \mathbb{R}^n$. Then the solution of (6.1) uniquely exists and satisfies:*

- when $c > c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad t > 0, \quad (6.2)$$

for $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$, and $\varepsilon_1 = \varepsilon_1(\tau)$ such that $0 < \varepsilon_1 < 1$ for $\tau > 0$ and $\varepsilon_1 = 1$ for $\tau = 0$

- when $c = c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \quad (6.3)$$

6.2 Fisher-KPP equation with nonlocal dispersion

For the equation (1.7), let $d(u) = u^2$, $b(u) = u$ and the delay $\tau = 0$, and take the limit of (1.7) as $\beta \rightarrow 0^+$, we get the classical Fisher-KPP equation with nonlocal dispersion without time-delay

$$\begin{cases} \frac{\partial u}{\partial t} - J * u + u = u(1 - u) \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \quad (6.4)$$

Then we have the existence of the monostable traveling waves and their stabilities from Theorem 2.1 and Theorem 2.2.

Theorem 6.3 (Traveling waves) *Let $J(x)$ satisfy (J_1) and (J_2) . For (6.4), there exists the minimal speed $c_* > 0$, such that*

- when $c \geq c_*$, the planar traveling waves $\phi(x \cdot \mathbf{e}_1 + ct)$ exist uniquely (up to a shift);
- when $c < c_*$, the planar traveling waves $\phi(x \cdot \mathbf{e}_1 + ct)$ do not exist;

Here $c_* := \lambda_*^{-1} \int_{\mathbb{R}} J_1(y_1) e^{-\lambda_* y_1} dy_1$, and $\lambda_* > 0$ is determined by $\int_{\mathbb{R}} (1 + \lambda_* y_1) J_1(y_1) e^{-\lambda_* y_1} dy_1 = 0$. When $c > c_*$, then $H_c(\lambda_*) < G_c(\lambda_*)$, where $H_c(\lambda_*) = 1$ and $G_c(\lambda_*) = c\lambda_* - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda_* y_1} dy_1 + 1$.

Theorem 6.4 (Stability of traveling waves) *Let $J(x)$ satisfy (J_1) and (J_2) , and the initial data be $u_0 - \phi \in C([- \tau, 0]; H_w^m(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ with $m > \frac{n}{2}$, and $u_- \leq u_0 \leq u_+$ for $x \in \mathbb{R}^n$. Then the solution of (6.4) uniquely exists and satisfies:*

- when $c > c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad t > 0, \quad (6.5)$$

for $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$;

- when $c = c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \quad (6.6)$$

6.3 Concluding Remark

Here we give a remark on the wave stability to the generalized equations with nonlocal dispersion. Let us consider a more general monostable equation with nonlocal dispersion

$$\begin{cases} \frac{\partial u}{\partial t} - J * u + u + d(u(t, x)) = F\left(\int_{\mathbb{R}^n} \kappa(y)b(u(t - \tau, x - y))dy\right), \\ u(s, x) = u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n, \end{cases} \quad (6.7)$$

where $J(x)$ satisfies (J_1) and (J_2) as mentioned before, and $F(\cdot)$, $d(u)$, $b(u)$ and $g(x)$ satisfy

(\mathcal{H}_1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = F(0) = 0$, $d(u_+) = F(b(u_+))$, $d \in C^2[0, u_+]$, $b \in C^2[0, u_+]$ and $F \in C^2[0, b(u_+)]$;

(\mathcal{H}_2) $F'(0)b'(0) > d'(0) \geq 0$ and $0 < F'(b(u_+))b'(u_+) < d'(u_+)$;

(\mathcal{H}_3) $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$ and $b''(u) \leq 0$ for $u \in [0, u_+]$;

(\mathcal{H}_4) $F'(u) \geq 0$ and $F''(u) \leq 0$ for $u \in [0, b(u_+)]$;

(\mathcal{H}_5) $\kappa(x)$ is a smooth, positive and radial kernel with $\int_{\mathbb{R}^n} \kappa(x)dx = 1$ and $\int_{\mathbb{R}^n} \kappa(x)e^{-\lambda x_1}dx < +\infty$ for all $\lambda > 0$.

Then, by a similar calculation, we can prove the existence of the traveling waves $\phi(x_1 + ct)$ for $c \geq c_*$, where $c_* > 0$ is a specified minimal wave speed, and that the noncritical traveling waves with $c > c_*$ are exponentially stable and the critical waves with $c = c_*$ are algebraically stable.

Theorem 6.5 (Traveling waves) *Assume that (J_1) -(J_2) and (\mathcal{H}_1) -(\mathcal{H}_5) hold. For (6.7), there exists the minimal speed $c_* > 0$, such that*

- *when $c \geq c_*$, the planar traveling waves $\phi(x \cdot \mathbf{e}_1 + ct)$ exist uniquely (up to a shift);*
- *when $c < c_*$, the planar traveling waves $\phi(x \cdot \mathbf{e}_1 + ct)$ do not exist;*

Here $c_* > 0$ and $\lambda_* = \lambda_*(c_*) > 0$ are determined by

$$\mathcal{H}_{c_*}(\lambda_*) = \mathcal{G}_{c_*}(\lambda_*) \quad \text{and} \quad \mathcal{H}'_{c_*}(\lambda_*) = \mathcal{G}'_{c_*}(\lambda_*),$$

where

$$\mathcal{H}_c(\lambda) = F'(0)b'(0) \int_{\mathbb{R}^n} e^{-\lambda y_1} \kappa(y)dy, \quad \mathcal{G}_c(\lambda) = c\lambda - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda y_1}dy_1 + 1 + d'(0).$$

When $c > c_*$, then

$$H_c(\lambda_*) < G_c(\lambda_*).$$

Theorem 6.6 (Stability of traveling waves) *Assume that (J_1) -(J_2) and (\mathcal{H}_1) -(\mathcal{H}_5) hold. Let the initial data be $u_0 - \phi \in C([-\tau, 0]; H_w^{m+1}(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ and $\partial_s(u_0 - \phi) \in L^1([-\tau, 0]; H_w^{m+1}(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ with $m > \frac{n}{2}$, and $u_- \leq u_0 \leq u_+$ for $x \in \mathbb{R}^n$. Then the solution of (6.7) uniquely exists and satisfies:*

- when $c > c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad t > 0, \quad (6.8)$$

for $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$, and $0 < \varepsilon_1 < 1$ for $\tau > 0$ and $\varepsilon_1 = 1$ for $\tau = 0$;

- when $c = c_*$, then

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \quad (6.9)$$

Acknowledgments The research of MM was supported in part by Natural Sciences and Engineering Research Council of Canada under the NSERC grant RGPIN 354724-08. The research of RH was supported in part by NNSFC (No. 11001103) and SRFDP (No. 200801831002).

References

- [1] D. S. ARONSEN AND H. F. WEINBERGER, *Multidimensional nonlinear diffusion arising in propagation genetics*, Adv. in Math., 30 (1978), pp. 33–76.
- [2] P. BATES, P. C. FIFE, X. REN AND X. WANG, *Traveling waves in a convolution model for phase transitions*, Arch. Rational Mech. Anal., 138 (1997), pp. 105–136.
- [3] M. BRAMSON, *Convergence of solutions of the Kolmogorov equation to traveling waves*, Mem. Amer. Math. Soc., 44 (285), (1983).
- [4] E. CHASSEIGNE, M. CHAVES AND J. ROSSI, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pure Appl., 86 (2006), pp. 271–291.
- [5] F. CHEN, *Almost periodic traveling waves of nonlocal evolution equations*, Nonlinear Anal., 50 (2002), pp. 807–838.
- [6] X. CHEN, *Existence, uniqueness and asymptotic stability of traveling waves in nonlocal evolution equations*, Adv. Differential Equations, 2 (1997), pp. 125–160.
- [7] X. CHEN, J.-S. GUO AND C.-C. WU, *Traveling waves in discrete periodic media for bistable dynamics*, Arch. Rational Mech. Anal., 189 (2008), pp. 189–236.
- [8] C. CORTAZAR, M. ELGUETA, J. D. ROSSI AND N. WOLANSKI, *How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems*, Arch. Rational Mech. Anal., 187 (2007), pp. 137–156.
- [9] J. COVILLE, *On uniqueness and monotonicity of solutions of non-local reaction-diffusion equation*, Annali di Matematica, 185 (2006), pp. 461–485.
- [10] J. COVILLE, J. DAVILA AND S. MARTINEZ, *Nonlocal anisotropic dispersal with monostable nonlinearity*, J. Differential Equations, 244 (2008), pp. 3080–3118.
- [11] J. COVILLE AND L. DUPAIGNE, *On a non-local equation arising in population dynamics*, Proc. Roy. Soc. Edinburgh, 137A (2007), pp. 727–755.

- [12] J. COVILLE AND L. DUPAIGNE, *Propagation speed of travelling fronts in non local reaction-diffusion equations*, Nonlinear Anal., 60 (2005), pp. 797–819.
- [13] P. C. FIFE, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomathematics, 28, Springer Verlag, 1979.
- [14] P. C. FIFE AND J. B. MCLEOD, *A phase plane discussion of convergence to travelling fronts for nonlinear diffusion*, Arch. Rational Mech. Anal., 75 (1980/81), pp. 281–314.
- [15] H. FREISTUHLER AND D. SERRE, L^1 STABILITY OF SHOCK WAVES IN SCALAR VISCOUS CONSERVATION LAWS, Comm. Pure Appl. Math., 51 (1998), pp. 291–301.
- [16] T. GALLAY, *Local stability of critical fronts in nonlinear parabolic partial differential equations*, Nonlinearity, 7 (1994), pp. 741–764.
- [17] J. GARCIA-MELIAN AND F. QUIROS, *Fujita exponents for evolution problems with nonlocal diffusion*, J. Evolution Equations, 10 (2010), pp. 147–161.
- [18] J. GOODMAN, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rational Mech. Anal., 95 (1986), pp. 325–344.
- [19] S. A. GOURLEY, *Travelling front solutions of a nonlocal Fisher equation*, J. Math. Biol., 41 (3) (2000), pp. 272–284.
- [20] S. A. GOURLEY AND J. WU, *Delayed non-local diffusive systems in biological invasion and disease spread*, in Nonlinear Dynamics and Evolution Equations (H. Brunner, X.-Q. Zhao and X. Zou, eds.), Fields Institute Communications, Amer. Math. Soc., 48 (2006), pp. 137–200.
- [21] F. HAMEL AND L. ROQUES, *Uniqueness and stability properties of monostable pulsating fronts*, J. European Math. Soc., to appear.
- [22] R. HUANG, *Stability of travelling fronts of the Fisher-KPP equation in \mathbb{R}^N* , Nonlinear Differential Equations Appl., 15 (2008), no. 4-5, pp. 599–622.
- [23] L. IGNAT AND J. D. ROSSI, *Decay estimates for nonlocal problems via energy methods*, J. Math. Pure Appl., 92 (2009), pp. 163–187.
- [24] L. IGNAT AND J. D. ROSSI, *A nonlocal convolution-diffusion equation*, J. Func. Anal., 251 (2007), pp. 399–437.
- [25] T. KAPITULA, *Multidimensional stability of planar traveling waves*, Trans. Amer. Math. Soc., 349 (1997), pp. 257–269.
- [26] S. KAWASHIMA AND A. MATSUMURA, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys., 101 (1985), pp. 97–127.
- [27] S. KAWASHIMA AND A. MATSUMURA, *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Comm. Pure Appl. Math., 47 (1994), pp. 1547–1569.

- [28] D. YA. KHUSAINOV, A. F. IVANOV AND I. V. KOVARZH, *Solution of one heat equation with delay*, Nonlinear Oscillations, 12 (2009), pp. 260–282.
- [29] K. KIRCHGASSNER, *On the nonlinear dynamics of travelling fronts*, J. Differential Equations, 96 (1992), pp. 256–278.
- [30] A. N. KOLMOGOROV, I. G. PETROVSKY AND N. S. PISKUNOV, *Etude de l' équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'Etat à Moscou, Série Internationale, Section A1, (1937), pp. 1–26.
- [31] K.-S. LAU, *On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piskounov*, J. Differential Equations, 59 (1985), pp. 44–70.
- [32] J.F. MALLORDY, J.M. ROQUEJOFFRE, *A parabolic equation of the KPP type in higher dimensions*, SIAM J. Math. Anal., 26 (1995), pp. 1–20.
- [33] H. MATANO, M. NARA, AND M. TANIGUCHI, *Stability of planar waves in the Allen-Cahn equation*, Comm. Partial Diff. Equations, 34 (2009), pp. 976–1002.
- [34] A. MATSUMURA AND M. MEI, *Convergence to travelling fronts of solutions of the p -system with viscosity in the presence of a boundary*, Arch. Ration. Mech. Anal., 146 (1999), pp. 1–22.
- [35] A. MATSUMURA AND K. NISHIHARA, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., 2 (1985), pp. 17–25.
- [36] A. MATSUMURA AND K. NISHIHARA, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys., 165 (1994), pp. 83–96.
- [37] M. MEI, C.-K. LIN, C.-T. LIN AND J. W.-H. SO, *Traveling wavefronts for time-delayed reaction-diffusion equation: (I) local nonlinearity*, J. Differential Equations, 247 (2009), pp. 495–510.
- [38] M. MEI, C.-K. LIN, C.-T. LIN AND J. W.-H. SO, *Traveling wavefronts for time-delayed reaction-diffusion equation: (II) nonlocal nonlinearity*, J. Differential Equations, 247 (2009), pp. 511–529.
- [39] M. MEI, C. OU, X.-Q. ZHAO, *Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations*, SIAM J. Math. Anal., 42 (2010), pp. 2762–2790.
- [40] , M. MEI, C. OU, X.-Q. ZHAO, *Exponential and algebraic stability of traveling wavefronts in periodic spatial-temporal environments*, to appear.
- [41] M. MEI AND Y. WANG, *Global stability of planar traveling waves for nonlocal Fisher-KPP type reaction-diffusion equations in multi-dimensional space*, submitted.
- [42] M. MEI AND Y. S. WONG, *Novel stability results for traveling wavefronts in an age-structured reaction-diffusion equations*, Math. Biosci. Engin., 6 (2009), pp. 743–752.

- [43] H. J. K. MOET, *A note on asymptotic behavior of solutions of the KPP equation*, SIAM J. Math. Anal., 10 (1979), pp. 728–732.
- [44] S. PAN, W.T. LI AND G. LIN, *Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay*, Nonlinear Anal., 72 (2010), pp. 3150–3158.
- [45] D. H. SATTINGER, *On the stability of waves of nonlinear parabolic systems*, Adv. Math., 22 (1976), pp. 312–355.
- [46] H. L. SMITH AND X.-Q. ZHAO, *Global asymptotic stability of traveling waves in delayed reaction-diffusion equations*, SIAM J. Math. Anal., 31 (2000), pp. 514–534.
- [47] J. W.-H. SO, J. WU AND X. ZOU, *A reaction-diffusion model for a single species with age structure: (I) Traveling wavefronts on unbounded domains*, Proc. Roy. Soc. London, Series A, 457 (2001), pp. 1841–1853.
- [48] A. SZEPESSY AND Z.-P. XIN, *Nonlinear stability of viscous shock waves*, Arch. Rational Mech. Anal., 122 (1993), pp. 53–103.
- [49] K. UCHIYAMA, *The behavior of solutions of some nonlinear diffusion equations for large time*, J. Math. Kyoto Univ., 18 (1978), pp. 453–508.
- [50] A. VOLPERT, VI. VOLPERT AND VL. VOLPERT, *Traveling Wave Solutions of Parabolic Systems*, Transl. Math. Monographs, 140, AMS, Providence, RI, 1994.
- [51] J. XIN, *Existence of multidimensional traveling waves in the transport of reactive solutes through periodic porous media*, Arch. Rational Mech. Anal., 128 (1994), pp. 75–103.
- [52] J. XIN, *Front propagation in heterogeneous media*, SIAM Rev., 42 (2000), pp. 161–230.
- [53] H. YAGISITA, *Existence and nonexistence of traveling waves for a nonlocal monostable equation*, Publ. Res. Inst. Math. Sci., 45 (2009), no. 4, pp. 925–953.
- [54] K. ZUMBRUN AND P. HOWARD, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana Univ. Math. J., 47 (1998), pp. 741–871.